TWENTIETH ANNUAL COLUMBUS INVITATIONAL MATHEMATICS TOURNAMENT

Sponsored by

THE COLUMBUS COLLEGE

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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The Columbus College Mathematics and Computer Science faculty welcome you to this year's tournament and to our campus. We hope that you will be successful on this test.

INSTRUCTIONS: This is a 90-minute, 50-problem, multiple-choice examination. There are five (5) possible responses to each question. You are to select the one (1) "best" answer to each. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the choice you have made on the test booklet. After you have worked all of the problems that you can work, carefully transfer your answers to the score sheet. Darken completely the blank corresponding to the letter of your response to each question. Mark your answers boldly with a No. 2 pencil. If you must change an answer, completely erase your first choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, -3 for each incorrect selection and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used for tie-breakers for individual awards. These problems, in the order in which they will be examined, are: 13, 17, 23, 36, and 40.

All pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. Your student identification number and your school number must be encoded correctly on your score sheet.

When you complete your test, bring your pencil, scratch paper and answer sheet to the Test Monitor. You may leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. **Please do not congregate outside the doors to the testing area.** You may keep your copy of the test. Your sponsor will have a copy of the answers.

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO DO SO

1. If $(.67)^{H} = .5$, find the value of $16(.67)^{3H}$.

a) 128 b) 64 c) 16 d) 8 e) 2

2. The equation $x^4 - 68x^2 + 256 = 0$ has four solutions. The absolute value of the sum of the four solutions is

a) 68 b) 0 c) 256 d) 20 e) 120

3. Two ordinary dice are tossed. What is the probability that the sum of the two dice is a number greater than 7?

a) 1 b) 5/12 c) 2/3 d) 4/9 e) 7/36

4. Find the approximate distance (to the nearest tenth) between the points of intersection of the graphs of $y = x^2 - 5x + 6$ and $y = x + 6$.

a) 72.0 b) 4.9 c) 9.0 d) 8.5 e) 3.5

5. The tangent line to the circle $x^2 + y^2 = 25$ at the point (3,4) intersects the x-axis at

a) 25/4 b) 19/3 c) 25 d) 25/3 e) -7/3

6. If the slope of a line is undefined, then the graph of the line

a) must pass through the origin.
b) is perpendicular to the graph of the line $y = x$.
c) is parallel to the graph of the line $y = 5$.
d) is perpendicular to the graph of the line $y = 5$.
e) is parallel to the graph of the line $y = 3x - 4$.

7. Which of the following sets of points is a set of collinear points?

a) {(8,-4),(3,7),(90,87)}   b) {(2,-3),(8,9),(4,-3)}

c) {(1,2),(2,3),(3,4)}   d) {(0,0),(0,-1),(7,9)}

e) none of the preceding

8. Find the length of the altitude of an equilateral triangle with sides each of length a.

a) $a\sqrt{3}/2$ b) $(a/3)^{1/2}$ c) a/2   d) $a\sqrt{2}$ e) $a/\sqrt{3}$

9. Suppose f is the function defined by $f(x) = x^2 + x$. Find the product of all solutions of the equation $f(3x - 1) = 12$.

a) -12   b) 2/3   c) -4/3   d) 8   e) -1/2

10. For all real non-zero numbers, $f(x) = 1 - 1/x$ and $g(x) = 1 - x$. If $h(x) = f(g(x))$, for what value of x does $h(x) = 8$?

a) 7   b) 2/7   c) 8/7   d) 3/8   e) 7/8

11. The minute hand of a clock is 2.5 inches long. Through what distance (in inches) does the tip of the minute hand move in 35 minutes?

a) $7\pi/6$ b) $7\pi/15$ c) $35\pi/24$ d) 5π e) $35\pi/12$

12. Find the greatest common factor of $48a^3bc$ and $72a^4bc^2$.

a) $144a^4bc^2$ b) $12a^4b^2c$ c) $12a^3bc$ d) 24abc

e) none of the preceding

13. Find the radius of a right circular cylinder of height 6 inches if the lateral surface area of the cylinder is the same as the surface area of a cube of edge 3 inches.

a) $9/(4\pi)$ in. b) 54 in. c) 27π in. d) $9/(2\pi)$ in.
e) $3/(\sqrt{\pi})$ in.

14. If $f(x) = 4x^3 - 3x^2 + 2x - 6$ and $g(x) = 7x^3 - 2x^2 + 3x - 5$, find the coefficient of x^2 in the product $f(x)g(x)$.

a) -5 b) -18 c) 33 d) 9 e) 27

15. How many four-letter "words" (sequences of letters with repetition of letters allowed) are there in which the vowels a,e,i,o,u do not appear as the second nor third letters?

a) 11025 b) 16900 c) 298116 d) 456976
e) 194481

16. Solve the following system of equations for x and y. Then find x + y.

$$\log_{10} x - \log_{10} y = -2$$
$$\log_{10} x + 2\log_{10} y = 1$$

a) 100 b) 10.1 c) -1 d) 1010 e) 6.67

17. Which of the following is <u>NOT</u> a sixth root of 1?

a) 1 b) $(-1 + i\sqrt{3})/2$ c) $(\sqrt{3} - i)/2$ d) -1

e) $((1 - i\sqrt{3})/2)^2$

18. A rectangle is capped with a semicircle. If the height of the rectangle is twice the base, which of the following expresses the area A of the figure as a function of the base x of the rectangle?

a) $2x^2 + \pi x$ b) $(\pi + 2)x^2$ c) $2x^2 + \pi x/2$ d) $2\pi x^2$
e) $(\pi/8 + 2)x^2$

19. If $0 < \theta < \pi$, and $x + \frac{1}{x} = 2\cos\theta$, find $x^3 + \frac{1}{x^3}$.

a) $2\cos\theta$ b) $8\cos 3\theta$ c) $2\cos 3\theta$ d) $8\cos^3\theta$
e) $2\cos^3 3\theta$

20. Express .001432 in scientific notation.

a) 1.4×10^4 b) 1.432×10^{-3} c) 1432×10^6
d) 1.432×10^3 e) $.1432 \times 10^{-2}$

21. Hypatia was born in October in A.D. 370 in Alexandria, Egypt. Some of her work was with conic sections. A right circular cylinder with radius 7 cm contains 1077 cm^3 of water. A rock is immersed in the water, and the water level rises 3 cm without any water being lost. What is the volume (in cm^3) of the rock?

a) 40π b) 49π c) 100π d) 147π e) 150π

22. The graph of the equation $y^2 - 6y - 6x + 39 = 0$ is a parabola. What is the focus of this parabola?

a) (13/2,3) b) (5,3) c) (7/2,3) d) (5,9/2)
e) (5,3/2)

23. The area of the small square in Figure 1 (see Figure Page at the end of the test) is one-third of the total area of the square. Calculate the ratio y/x.

a) $\sqrt{3} - 1$ b) $\frac{\sqrt{3} + 1}{2}$ c) 1 d) $\frac{1}{3}$ e) $\frac{\sqrt{3}}{2}$

24. A town has a population of 300,000 and an annual growth rate of 4.5 percent. Find the population after 11 years.

a) 492,149 b) 300,495 c) 486,856 d) 500,000
e) 313,500

25. One-fourth is the same part of 1/3 as 1/2 is of what number?

a) $\frac{1}{6}$ b) $\frac{3}{4}$ c) $\frac{2}{3}$ d) $\frac{7}{12}$ e) $\frac{3}{2}$

26. Tao has a fenced-in circular dog pen with an area of 78.5 m^2. He wants to increase the area to 100 m^2 (keeping it circular). To the nearest meter, how much extra fencing will he need?

a) 3 b) 4 c) 5 d) 6 e) 7

27. Let f be an even function (meaning $f(-x) = f(x)$) and let g be an odd function (meaning $g(-x) = -g(x)$), both functions having the whole real line as their domains. Which of the following is not an odd function?

a) $f(x)g(x)$ b) $f(x) + g(x)$ c) $g(g(x))$ d) $\frac{f(x)}{g(x)}$

e) $(g(x))^3$

28. The arithmetic mean of x and 4x is 20. What is x?

a) 4 b) 5 c) 8 d) 10 e) cannot be determined

29. How many rectangles are there with integral sides, in which the number for the area is equal to the number for the perimeter?

a) 2 b) 3 c) 4 d) 5 e) 6

30. Given the right triangle in Figure 2 (see Figure Page at the end of the test), write θ explicitly in terms of x.

a) $\sin(x/3)$ b) $\sin^{-1}(x/3)$ c) $\cot(3/x)$

d) $\cos^{-1}(3/x)$ e) $1/(\cos(3/x))$

31. Which of the following sets of whole numbers has the largest average?

a) Multiples of 2 between 1 and 101
b) Multiples of 3 between 1 and 101
c) Multiples of 4 between 1 and 101
d) Multiples of 5 between 1 and 101
e) Multiples of 6 between 1 and 101

32. As shown in Figure 3 (see Figure Page at the end of the test), a triangle with side lengths DB = EB = 1 is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral ADEC is

a) 6 b) 6.5 c) 7 d) 7.5 e) 8

33. Let c be a constant. The simultaneous equations

$$x - y = 2$$
$$cx + y = 3$$

have a solution (x,y) inside Quadrant I if and only if

a) $c = -1$ b) $c > -1$ c) $c < 1.5$ d) $0 < c < 1.5$
e) $-1 < c < 1.5$

34. What digit must A be in order for the number 5399A523136 to be divisible by 8?

a) 2 b) 3 c) 5
d) No digit 0-9 will work e) Any digit 0-9 will work

35. A binary operator * is defined on the positive integers by the rule $a*b = b^a$. Which of the following properties hold for *?

I) a*b = b*a for all positive integers a and b.
II) (a*b)*c = a*(b*c) for all positive integers a, b, and c.
III) There exists a unique positive integer k such that a*k = k*a = a for every positive integer a.
IV) The set of positive integers is closed under *, i.e., a*b is a positive integer for all positive integers a and b.

a) IV only b) I and III c) II only d) II and IV
e) none of the properties hold

36. Experimentation and theory have shown that an approximate rule for atmospheric pressure at altitudes less than 80 km is the following: Standard atmospheric pressure, 1035 grams per square centimeter, is halved for each 5.8 km of vertical ascent. Find the approximate altitude at which the atmospheric pressure is 20 percent of standard atmospheric pressure.

a) 10.5 km b) 13.5 km c) 12.8 km d) 13.8 km
e) 14.7 km

37. Two telephone poles were erected 100 feet apart. One pole is 20 feet high and the other is 80 feet high. If cables are connected from the top of each pole to the base of the opposite pole (with no slack in the cables), how high is the point of intersection of the two cables?

a) 15 ft b) 20 ft c) 16 ft d) 12 ft
e) none of the preceding

38. Find the product of the determinants of the given matrix A and its inverse A^{-1}.

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) 0 b) 1 c) 2 d) 7 e) Cannot be determined

39. A sequence of equilateral triangles is constructed as follows: The first triangle has sides 2 meters in length. To get the second triangle, midpoints of the sides of the original triangle are connected. If this process could be continued indefinitely, what would be the sum of the perimeters of all the triangles?

a) 6 b) ∞ c) 12 d) 11.25 e) cannot be determined

40. A point moves in the xy-plane so as to be always equidistant from the points (-2,-6) and (4,-2). What is the equation of its path?

a) $y = -\frac{3}{2}x - \frac{5}{2}$ b) $y = \frac{2}{3}x - \frac{5}{2}$ c) $y = -\frac{3}{2}x$

d) $(x+2)^2 + (y+6)^2 = 52$ e) $y = 2x^2 - 5x + 3$

41. Find the area of the shaded region of the right triangle ABC shown in Figure 4 (see Figure Page at the end of the test), given that the arc is circular, where AC is the radius of the circular sector.

a) 7500 b) 2741 c) 4914 d) 2586 e) 4759

42. A number is called "cute" if it has exactly four distinct positive integer divisors. What percent of the first twenty-five positive integers are cute?

a) 6 b) 7 c) 24 d) 28 e) 32

43. The graph of the equation $x^3 - 2x^2y + xy^2 - 2y^3 = 0$ is precisely the graph of

a) $x^2 + y^2 = 0$ b) $x = y$ c) $y = 2x^2 - x$ d) $x = y^3$

e) $x = 2y$

44. Find the maximum product of two positive real numbers whose sum is 45.

a) 506.25 b) 500 c) 400 d) 625 e) 506

45. The semicircles in Figure 5 (see Figure Page at the end of the test) have sides of the square as diameters. If a side of the square has measure 4 units, what is the area of the shaded region?

a) $8 - \pi$ b) $12 - 2\pi$ c) $16 - 11\pi/3$ d) $16 - 4\pi$

e) $4 + 2\pi$

46. An urn contains 10 balls: 5 are white, 3 are red, and 2 are black. Three balls are drawn at random, with replacement, from the urn. What is the probability that all 3 balls are different colors?

a) .03 b) .09 c) .18 d) .40 e) .84

47. Gabrielle Emilie du Chatelet is well known for her translation and analysis of Newton's masterpiece, Principia. The year of her birth is a four-digit number in which the first two digits are the seventh prime number, the third number is the additive identity, and the last digit is the second composite number. In what year was she born?

a) 1304 b) 1316 c) 1706 d) 1714 e) 1716

48. Two chords intersect within a circle at point P as indicated in Figure 6 (see Figure Page at the end of the test). Find the area of $\triangle PCB$ if the area of $\triangle PAD = 8$ and the ratio of AP to PC is 2 to 5.

a) 40 b) 50 c) 100 d) 125 e) 20

49. The dart board shown in Figure 7 (see Figure Page at the end of the test) has a 7" radius with each region of the dart board 2" wide, except the bull's eye, which has a 1" radius. If a dart is thrown at random and is guaranteed to hit the board and it is no more likley that the dart will hit any one point on the board than any other, what is the probability that it will hit the ring worth 50 points?

a) .20 b) .25 c) .29 d) .33 e) .39

50. Insert three fractions between 1/4 and 1/2 so that the five fractions form an arithmetic sequence. What is the sum, expressed as a common fraction, of these three new fractions?

a) 9/8 b) 11/8 c) 1/2 d) 15/8 e) 3/4

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FIGURE PAGE

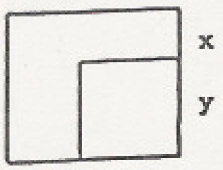

Figure 1
prob. #23

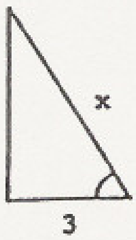

Figure 2
prob. #30

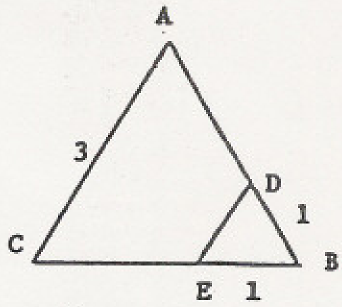

Figure 3
prob. #32

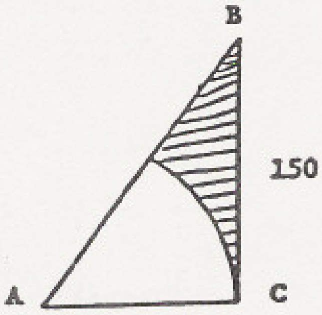

Figure 4
prob. #41

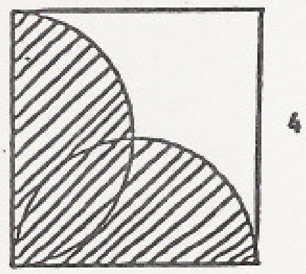

Figure 5
prob. #45

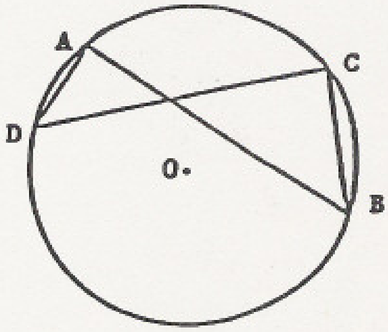

Figure 6
prob. #48

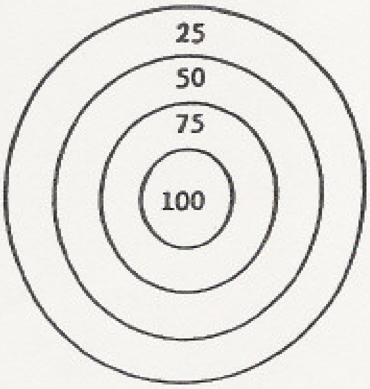

Figure 7
prob. #49