Thirty-second Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
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The Mathematics Department at Columbus State University welcomes you to our campus and to this year's tournament. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200. Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 8, 12, 18, 24, 25, 32, 36, and 43.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.** When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) What is the sum of all the solutions of the equation $x^3 - 78x^2 + 1155x - 2006 = 0$?

A) 74 B) 44 C) -40 D) 78 E) -78

2) The sale price of a printer is 30% off its original price of \$100. An employee gets an additional 20% off this sale price. What would an employee pay for this printer if it was purchased on a tax-free day in Georgia?

A) \$34 B) \$55 C) \$45 D) \$57 E) \$56

3) Which of the following is not equal to $2^{2^{2^2}}$?

A) 4^{2^3} B) 8^{2^4} C) 16^{2^2} D) 2^{4^2} E) 2^{2^4}

4) Dr. Henning's lecture was so boring that one-half of the audience left after a few minutes. Five minutes later, one-third of the remaining audience left. Ten minutes later, one-fourth of those remaining left, leaving only nine people in the audience. How many people were in the audience at the beginning of the lecture?

A) 21 B) 36 C) 72 D) 144 E) 216

5) Solve for x the equation $\frac{1}{x-3} + \frac{1}{x+3} = \frac{6}{x^2-9}$.

A) -3 B) 3 C) -3 and 3 D) The equation has no solutions

E) None of the above

6) A pole rests against a vertical wall 20 feet above the ground, while the base is 5 feet from the wall. If the pole is moved so that it rests 16 feet above the ground, how many feet is the base from the wall?

A) 13 ft B) $\sqrt{15}$ ft C) 9 ft D) $5\sqrt{15}$ ft E) $5+\sqrt{15}$ ft

7) Which is the set of real numbers for which the identity $\sqrt{x^2+3x-4} = \sqrt{x+4}\sqrt{x-1}$ is true?

A) $(1,\infty)$ B) $[1,\infty)$ C) $(-\infty,-4]$ D) $[-4,\infty)$

E) $(-\infty,-4]\cup[1,\infty)$

8) * Find all solutions of the inequality $x^2 - 3x < |4x - 6|$.

A) $1 < x < 6$ B) $-3 < x < 2$ C) $-3 < x < 6$

D) $x < -3$ or $6 < x$ E) $-3 < x < 1$

9) Find the sum of all real solutions of the equation $\sqrt[4]{7x^2 - 6} = x$.

A) $1 + \sqrt{6}$ B) 0 C) 2 D) $2(1 + \sqrt{6})$ E) None of these

10) What is the number of pairs (x, y) of integers satisfying the inequality $|x| + |y| \le 10$?

A) 261 B) 231 C) 221 D) 225 E) 441

11) Which of the following is the largest value of δ that guarantees that for all x satisfying $-\delta \le x - 2 \le \delta$, we will have $|x^2 - 4| \le 0.25$?

A) 3.936491673 B) 0.0635083269 C) 0.06155281281

D) 4.061552813 E) 4.0635083269

12) * Two students attempted to solve a quadratic equation $2x^2 + ax + b = 0$. Although both students did the work correctly, the first could not read the coefficient in the middle term, made up a value for it and obtained the solution set $\{-4, 3\}$. The second miscopied the constant term and obtained the solution set $\{5, -4\}$. What is the correct solution set?

A) $\{4, -3\}$ B) $\{2, -3\}$ C) $\{-2, 3\}$ D) $\{-5, 3\}$

E) Not enough information.

13) If $\log_a x = 4$, find $\log_{1/a} x$.

A) $-\frac{1}{4}$ B) $\frac{1}{4}$ C) 4 D) -4 E) $\frac{1}{2}$

14) Solve for a in the following equation $a^{2+\log_a 4} = a^2 + 2$.

A) $a = \sqrt{\frac{2}{3}}$ B) $a = \pm\sqrt{\frac{2}{3}}$ C) $a = \sqrt{2}$ D) $a = \frac{2}{3}$ E) $a = 4$

15) For each real number x, we define $\lceil x \rceil$ to be the least integer which is greater than or equal to x. For example, $\lceil 3.2 \rceil = 4$ and $\lceil 4 \rceil = 4$. If x and y are real numbers for which $\left\lceil \sqrt[3]{x} \right\rceil = 3$ and $\left\lceil \sqrt[3]{y} \right\rceil = 4$, what is the smallest possible value of $\lceil x + y \rceil$?

A) 59 B) 92 C) 35 D) 36 E) 25

16) A palindromic number is a natural number that is the same when written forwards or backwards. For instance, 323 is palindromic number. How many three-digit palindromic numbers are there?

A) 90 B) 81 C) 100 D) 99 E) 111

17) Find the sum of all the three-digit palindromic numbers.

A) 49500 B) 50400 C) 45900 D) 49950 E) 44955

18) * A lumberjack has $4n + 110$ logs in a pile consisting of n layers. Each layer has two more logs than the layer directly above it. If the top layer has six logs, how many layers are there?

A) -11 B) 11 C) 8 D) 9 E) 10

19) For how many integers n between 1 and 100, do the numbers $n^2 + 2$ and $n + 1$ have a common factor greater than 1?

A) 35 B) 3 C) 25 D) 32 E) 33

20) Find the value of n for which $\sum_{i=1}^{n} \frac{2006}{i(i+1)} = 2005$.

A) 2005 B) 2006 C) 2007 D) 2004

E) None of these

21) Find the remainder of $2^{2006}+1$ when divided by 3.

A) 0 B) 1 C) 2 D) -1 E) $\frac{1}{3}$

22) Find the solution set of the equation $\left(\frac{x}{x-1}\right)^2 = -6\left(\frac{x}{x-1}\right)+7$.

A) $\{1/8, 0\}$ B) $\{1/8\}$ C) $\{-1/8\}$ D) $\{7/8\}$ E) $\{7/8, 0\}$

23) At sunrise, two drivers started driving (in straight line) towards each other, each at their own constant speed. One started at point A and went towards point B while the other one started at point B and went towards point A. They met at noon but did not stop; each continued driving maintaining their speed and direction. The first driver arrived at point B at 4:00 pm and the other one arrived at point A at 9:00 pm. At what time did the sun rise that day?

A) 5 am B) 6 am C) 6:30 am D) 7:00 am

E) Not enough information.

24) * A bag contains 16 chips that are either red or blue. It is known that the probability of selecting (without replacement) a red chip and then blue chip (in that order) is $1/4$, and that the probability of selecting two blue ones is $1/8$. How many red chips are there in the bag?

A) 10 B) 12 C) 6 D) 5 E) 7

25) * Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 a.m. If a certain passenger arrives at the station at a (uniform) random time between 7 and 8 a.m. and then gets on the first train that arrives, what proportion of time does the passenger go to destination A?

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $\frac{1}{2}$ D) $\frac{3}{4}$ E) $\frac{1}{4}$

26) A standard Virginia license plate is of the form CBE-5678, three letters followed by four numbers. How many such plates have exactly two fours and at least one H?

A) 1,024,002 B) 303,750 C) 316,062 D) 911,250 E) 948,186

27) What is the value of the sum of the first 101 positive integers if we exclude all multiples of 3?

A) 3465 B) 3366 C) 3367 D) 3468

E) None of these

28) Find x if $3^{27^{(669x-1)}} = 27^{3^{2005x}}$.

A) -1 B) 1 C) -2 D) 2

E) None of these

29) What is the number of integers between 1 and 2006 (inclusive) that are not divisible by 2 or 5?

A) 1204 B) 801 C) 802 D) 601

E) None of these

30) There are three distinct lines and two distinct circles. What is the largest number of points in which at least two of these figures can intersect?

A) 14 B) 15 C) 16 D) 17 E) 18

31) A square-walled city of unknown dimension has four gates, one at the center of each side. A tree stands outside the city 30 meters north of the northern gate. One must walk 30 meters southward from the south gate and then turn west and walk 1485 meters before one can see the tree. What are the dimensions of the city?

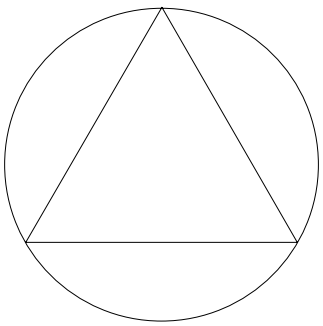
A) 267 meters $\times$ 267 meters B) 284 meters $\times$ 284 meters C) 270 meters $\times$ 270 meters

D) 250 meters $\times$ 250 meters E) None of these

32) * An equilateral triangle is inscribed in a unit circle. What is the area of the equilateral triangle?

A) $\frac{3\sqrt{3}}{4}$ B) $\frac{3\sqrt{3}}{2}$ C) $\frac{3}{4}$

D) $\frac{3\sqrt{2}}{4}$ E) $\frac{3\sqrt{2}}{2}$

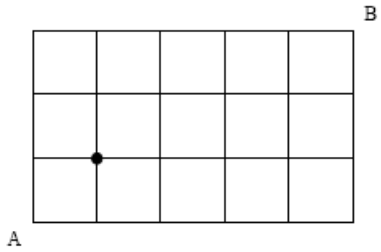


33) A goat is tethered to a corner of a square building into which there is no access for the goats. The building is 40 feet per side; the tether is 60 feet long. Assuming that the entire area surrounding the building is covered with grass, what is the total grazing area for the goat?

A) 2900π square feet B) 1800π square feet C) 2700π square feet

D) $(3600\pi - 2500)$ square feet E) 3000π square feet

34) How many different routes are there from A to B in the following figure if we can travel only up or to the right and we cannot go through the node point indicated by $\bullet$?



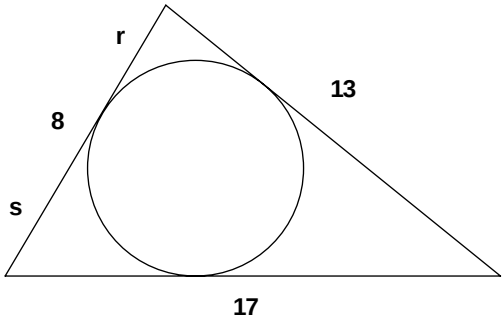
A) 41 B) 26 C) 56

D) $2^8 - 4$ E) None of these

35) A circle is inscribed in a triangle with sides that measure 8, 13, and 17. Let the point of tangency on the side of length 8 divide the side into segments of length r and s, where $r < s$. Find the ratio $\frac{r}{s}$.

A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) $\frac{1}{2}$

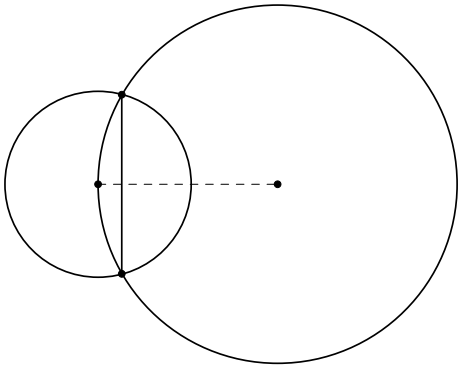
D) $\frac{2}{3}$ E) $\frac{3}{4}$



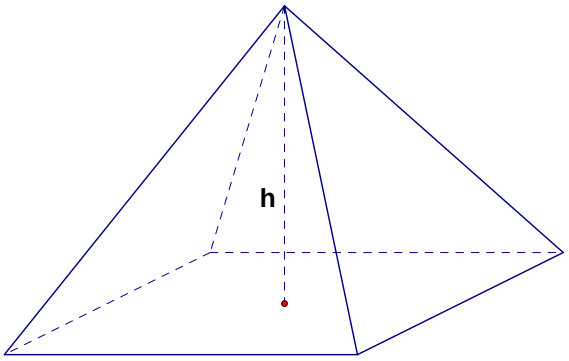
36) * Given two circles, one of radius R and the other of radius $2R$, assume that the larger circle passes through the center of the smaller. Find the length of the line connecting the two points of intersection of the two circles.

A) $\frac{\sqrt{15}}{2}R$ B) $\frac{\sqrt{3}}{2}R$ C) $\frac{\sqrt{3}}{4}R$

D) $\frac{1}{4}R$ E) $\frac{\sqrt{15}}{4}R$



37) The pyramid of square base shown in the figure has a height of $h = 1$ foot. If all edge lengths are the same, find the volume V of the pyramid.



A) $V = \frac{\sqrt{2}}{3}$ B) $V = \frac{3}{2}$

C) $V = \frac{2}{3}$ D) $V = 2$

E) $V = \sqrt{2}$

38) You are about to leave for school in the morning and discover you do not have your glasses. You know the following statements are true.

I. If my glasses are on the kitchen table, then I saw them at breakfast.
II. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
III. If I was reading the newspaper in the living room, then my glasses are on the coffee table.
IV. I did not see my glasses at breakfast.
V. If I was reading my book in bed, then my glasses are on the bed table.
VI. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Where are the glasses?

A) On the kitchen table B) On the coffee table C) On the bed table

D) In the living room E) not enough information to answer the question.

39) Eight identical sheets of paper have been placed overlapping. The last sheet to be placed was H because it is shown completely. List the eight pieces in the order placed, ending with H.

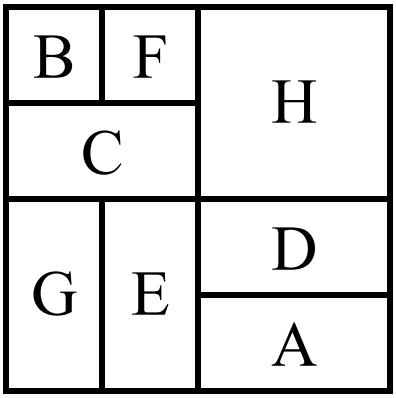
A) The order is B, F, C, E, G, A, D, H

B) The order is G, E, A, D, B, F, C, H

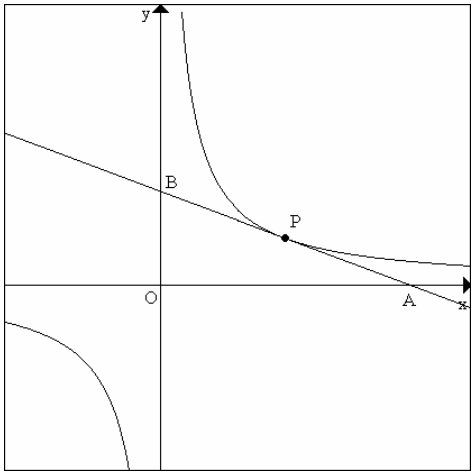
C) The order is F, B, C, G, E, D, A, H

D) The order is B, F, C, G, E, A, D, H

E) The order cannot be determined.



40) Consider the hyperbola H given by the equation $xy = 6$. Observe that the point $P(3,2)$ lies on H. Let L be the line tangent to H at $P(3,2)$. Then L intersects the positive x-axis at a point A and the positive y-axis at a point B. Find the area of the triangle $\triangle AOB$.



A) 12 B) 6 C) 10 D) 18

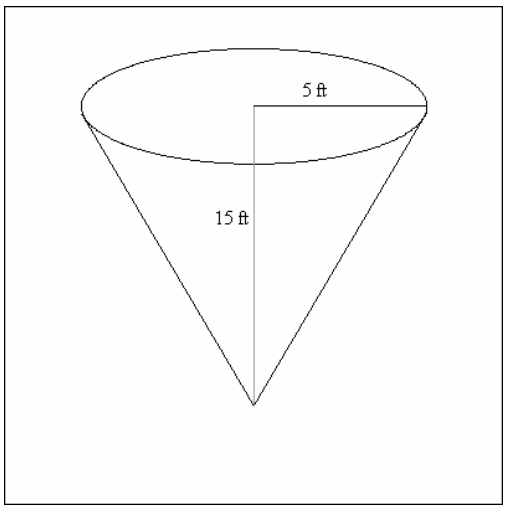
E) 24

41) Which of the following defines the ratio of the surface area to the volume of a right circular cone with radius r and height h?

A) $\dfrac{3\left(r+\sqrt{r^2+h^2}\right)}{rh}$ B) $\dfrac{3\left(\pi r^2+\sqrt{r^2+h^2}\right)}{\pi rh}$ C) $\dfrac{3\left(r^2+\pi\sqrt{r^2+h^2}\right)}{\pi rh}$

D) $\dfrac{\pi r+\sqrt{r^2+h^2}}{3\pi rh}$ E) $\dfrac{3\left(r^2+\sqrt{r^2+h^2}\right)}{\pi rh}$

42) A tank that has the form of an inverted cone (see figure on the right) of radius 5 ft and height 15 ft contains 45 ft^3 of water. If at time $t = 0$ minutes a pump starts pumping water into the tank at a rate of 3 ft^3/min, which of the following is the least amount of time that will guarantee the tank is full?



A) 115 min. B) 115.5 min.

C) 115.9 min. D) 125.9 min.

E) 130.9 min.

43) * There are 120 permutations of the number 43512. Suppose that we arrange them in increasing order from the first one, 12345 to the largest one, 54321. What permutation will be the 55th place?

A) 32145 B) 31542 C) 34521 D) 35421 E) 32154

44) Which of the following functions is equal to $f(x) = \sin^{-1}(\sin x)$ for all $-\infty < x < \infty$?

A) $-x$ B) x C) $\frac{1}{\sin x}$ D) $\frac{1}{\sin(\sin x)}$ E) None of these

45) Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined recursively by $f(1) = 1$ and $f(n+1) = 2f(n) + 1$ for all $n \geq 1$. Find the value of $f(1) + f(2) + \cdots\cdots + f(10)$.

A) 2006 B) 2008 C) 2010 D) 2036 E) 2048

46) Find the value of $\tan^{-1}(\tan\theta)$ where $\frac{3}{2}\pi < \theta < \frac{5}{2}\pi$.

A) $\theta - 3\pi$ B) $\theta - \frac{5\pi}{2}$ C) $\theta - 2\pi$ D) $\theta - \frac{3\pi}{2}$ E) θ

47) If $0 < b < c$, find the value of $\cos\left(\sin^{-1}\left(\frac{b}{c}\right)\right)$.

A) $\frac{\sqrt{c^2+b^2}}{c}$ B) $\frac{c}{\sqrt{c^2+b^2}}$ C) $\frac{\sqrt{c^2-b^2}}{c}$ D) $\frac{c}{\sqrt{c^2-b^2}}$

E) None of these

48) Find the area of a triangle having sides of length 6 and 4, which subtend an angle of 60^o.

A) $\frac{\sqrt{3}}{3}$ B) 3 C) 6 D) $6\sqrt{3}$ E) $3\sqrt{3}$

49) Find all real values of a for which the following linear system has no solutions.

$$\begin{aligned} x_1 + \quad & x_2 = 3 \\ x_1 + (a^2 - 8)x_2 &= a \end{aligned}$$

A) $a = \pm 3$ B) $a = \pm\sqrt{8}$ C) $a = -3$ D) $a = 3$

E) The system has at least one solution for any value of a.

50) Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ and suppose that $\det(A) = 2$. Find the $\det(B)$, where $B = \begin{bmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{bmatrix}$.

A) $\det(B) = -2$ B) $\det(B) = \frac{1}{2}$ C) $\det(B) = -8$

D) $\det(B) = -16$ E) $\det(B) = 8$

51) There are two positive solutions to the equation $\log_{2x} 2 + \log_4 2x = -\frac{3}{2}$. What is the product of the two solutions?

A) $1/21$ B) $3/21$ C) $1/32$ D) $1/8$ E) 2

52) Find, wherever it is defined, the derivative of $f(x) = \sin^{-1}(\sin x)$.

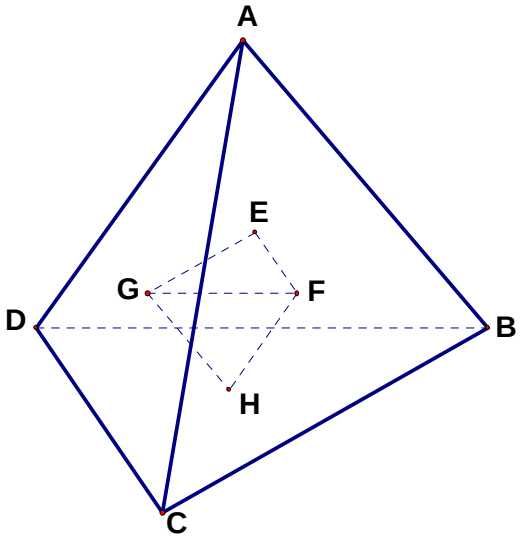
A) -1 B) 1 C) $\frac{\cos x}{|\cos x|}$ D) $\frac{\sin x}{|\cos x|}$ E) None of these

53) Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ and suppose that $\det(A) = 6$. Find the $\det(B)$, where $B = \begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix}$

A) $\det(B) = 0$ B) $\det(B) = -12$ C) $\det(B) = 12$

D) $\det(B) = -6$ E) $\det(B) = 6$

54) The figure (not drawn at scale) shows a regular tetrahedron $ABCD$ and its dual, obtained by joining the centers of all of its faces. Find the ratio $\frac{Volume\ ABCD}{Volume\ EFGH}$ of their volumes



A) 27 B) $9\sqrt{3}$ C) 15

D) $3\sqrt{3}$ E) $26\sqrt{3}$