Thirty-first Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
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The Mathematics Department at Columbus State University welcomes you to our campus and to this year's tournament. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200. Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 16, 23, 25, 32, 36, 38, 42 and 48.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.** When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) A gumball machine contains 2005 gumballs. The number the gumballs of each color is equally divided among red, orange, yellow, green, and blue. How many gumballs must a customer purchase to guarantee that he/she receives 3 blue gumballs?

A) 3 B) 9 C) 11 D) 803 E) 1607

2) Simplify the expression $(\sqrt[3]{a})^{\log_3 27}$.

A) 1 B) a C) 2 D) 3 E) $3a$

3) Which of the following usual operations on the real numbers $+, -, \div, \bullet$ (addition, subtraction, division, and multiplication, respectively) is/are associative?

A) $+$ B) $+, -$ C) $+, \bullet$ D) $+, \div, \bullet$ E) $+, -, \div, \bullet$

4) Simplify $\left(-\frac{1}{27}\right)^{-2/3}$.

A) $-\frac{1}{3}$ B) $\frac{1}{3}$ C -3 D) 9 E) $3\sqrt{-1}$

5) If the reciprocal of $x+a$ is $x-a$ and a is a real number, what is the value of x?

A) 0 B) a C) $-a$ D) a or $-a$ E) none of these

6) If $a>0$, what is the value of the product $(\log_a 3)(\log_3 5)(\log_5 a^4)$?

A) 1 B) 2 C) 3 D) 4 E) 5

7) An object, moving on a straight line, travels $\frac{a}{6}$ feet in t seconds. How many yards does it travel in 3 minutes at this rate?

A) $\frac{10a}{t}$ yards B) $\frac{20a}{t}$ yards C) $\frac{180a}{t}$ yards D) $\frac{a}{2t}$ yards E) $\frac{30a}{t}$ yards

8) Which of the following quadrants contain the set of points (x, y) that satisfy both inequalities $y > x$ and $y > 2 - x$?

A) I, III B) I, IV C) II, III D) I, II E) III, IV

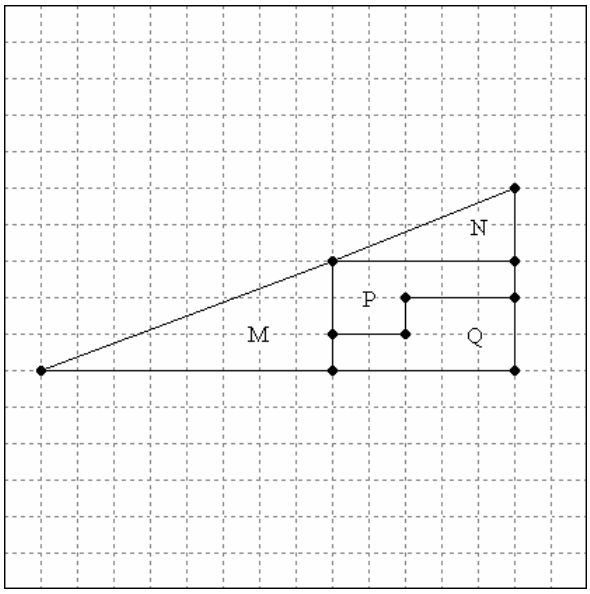
9) Solve the equation $\frac{1}{2}\left(\frac{2004}{x-2004}+\frac{2006}{x-2006}\right)=\frac{2005}{x-4010}$ for x.

A) 2001 B) 2002 C) 2003 D) 2004 E) 2005

10) Solve the equation $x^{\log x}=\frac{x^2}{10}$ for x.

A) 1 B) 10 C) 20 D) 1 and 2 E) -1 and 1

11) The figure on the right shows a Cartesian grid where each square is 1 inch by 1 inch. Ten points were selected on the grid (indicated by $\bullet$) and joined by segments to build the four regions M, N, P, Q. If S is the sum of the areas of the four regions M, N, P, Q, find S.



A) $S = 31\,in^2$ B) $S = 65\,in^2$

C) $S = 32.5\,in^2$ D) $S = 32\,in^2$

E) $S = 64\,in^2$

12) Find an equation of the line that passes through the midpoint of the segment joining the points $P = (3,7)$, $Q = (9,3)$ and is perpendicular to the segment.

A) $y-5=\frac{3}{2}(x-6)$ B) $y-6=-\frac{3}{2}(x-5)$ C) $y-5=-\frac{2}{3}(x-6)$

D) $y-6=-\frac{2}{3}(x-5)$ E) $y-6=\frac{3}{2}(x-5)$

13) A gear with 101 teeth meshes with a second wheel which has 401 teeth. A motor drives the first wheel at the rate of 2005 revolutions per minute. What is the speed of the second wheel in revolutions per minute?

A) 25.25 B) 505 C) 505π D) 1010 E) 1010π

14) Find all real solutions of the equation $\sqrt{3x+7} - x = 2$.

A) 1.302775638 and -2.302775638 B) 1.302775638 C) $-\frac{1}{2}+\frac{1}{2}\sqrt{13}$

D) $-\frac{1}{2}+\frac{1}{2}\sqrt{13}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{13}$ E) None of these

15) What is the product of all the rational solutions of the equation $x^4 - 2x^3 - 3x^2 + 2x + 2 = 0$?

A) 2 B) -2 C) 1 D) -1 E) 0

16) * How many ordered pairs of positive integers are solutions of $2^{2x} - 3^{2y} = 2005$?

A) 0 B) 1 C) 2 D) 3 E) more than three

17) A beam of length 30 feet stands against a vertical wall. The upper end has slipped down 6 feet. How far did the lower end move?

A) 18 feet B) 10 feet C) 20 feet D) 17 feet E) 15 feet

18) A store sells cashews for \$6 per pound and peanuts for \$2.50 per pound. The manager decides to mix 40 pounds of peanuts with some cashews and sell the mixture for \$4 per pound. How many pounds of cashews should be mixed with the peanuts so that the mixture will produce the same revenue as would selling the nuts separately?

A) 30 B) 53.4 C) 0 D) 54

E) None of these

19) If $\cos\theta\cos 2\theta = \frac{1}{4}$ and $0^\circ < \theta < 90^\circ$, then θ is in which interval?

A) $0^\circ < \theta < 10^\circ$ B) $10^\circ < \theta < 20^\circ$ C) $20^\circ < \theta < 30^\circ$

D) $30^\circ < \theta < 40^\circ$ E) $40^\circ < \theta < 50^\circ$

20) The sides of a triangular garden with area 200 square meters are in proportion 3:2:2. Find the length of each side in meters to nearest hundredth of a meter.

A) 30.12, 20.08, 20.08 B) 60.24, 40.16, 40.16 C) 60, 40, 40

D) 30, 20, 20 E) 15.06, 10.04, 10.04

21) The sum of 401 consecutive integers $a_1, a_2, a_3, \ldots, a_{401}$ is equal to 2005. What is a_{200}?

A) 4 B) -196 C) 5 D) 3 E) 206

22) What is the coefficient of x^0 in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^{2005}$ in terms of powers of x?

A) $\frac{2005!}{802!1203!}$ B) $\frac{2005!}{1203!}$ C) $\frac{2005!}{802!}$ D) 1

E) There is no x^0 in the expansion of the given expression.

23) * Find the remainder of $2^{2005}+1$ when divided by 11.

A) 0 B) 3 C) 7 D) 8 E) 10

24) Find all real solutions of the inequality $\left|x^2 - 3x\right| \le 4x - 6$.

A) $2 \le x \le 6$ B) $2 < x < 6$ C) $x \le 2$ or $6 \le x$

D) $x < 2$ or $6 < x$ E) None of these

25) * How many zeroes are at the end of the expansion of the number $2005!$?

A) 480 B) 420 C) 401 D) 500 E) 501

26) A tank contains 4000 liters of pure water. Brine that contains 40 grams of salt per liter of water is pumped into the tank at a rate of 20 liters per minute. Find the concentration $C(t)$ of salt in the water at t minutes in grams per liter.

A) $C(t)=\dfrac{40t}{200+t}$ B) $C(t)=\dfrac{200+t}{40t}$ C) $C(t)=\dfrac{20t}{200+t}$

D) $C(t)=\dfrac{40t}{4000+t}$ E) $C(t)=40t$

27) Given a number of the form $a+b\sqrt{2}$, where a,b are rational numbers with $a\neq 0$ or $b\neq 0$, there exists a number $p+q\sqrt{2}$, with p,q rational numbers, such that $\left(a+b\sqrt{2}\right)\left(p+q\sqrt{2}\right)=1$. What is the value of q ?

A) $\dfrac{a}{a^2-2b^2}$ B) $\dfrac{a}{a^2+2b^2}$ C) $\dfrac{-b}{a^2-2b^2}$ D) $\dfrac{-b}{a^2+2b^2}$ E) $\dfrac{-b}{a^2-4b^2}$

28) An infinite sequence $a_0,a_1,a_2,\ldots,a_n,\ldots$ of positive integers satisfies the following three conditions:

$$a_0=3,\ a_1=7, \text{ and } a_n=3a_{n-1}-2a_{n-2} \text{ for all } n\geq 2 .$$

Which of the following gives a_n for all $n\geq 0$?

A) $a_n=4n+3$ B) $a_n=2n+1$ C) $a_n=2^{n+2}-1$ D) $a_n=2^{2n+1}-1$

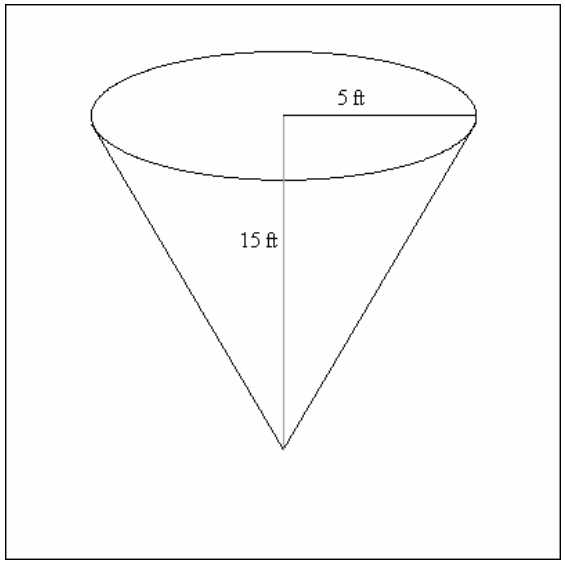
E) $a_n=2(3)^n+1$

29) A square-walled city of unknown dimension has four gates, one at the center of each side. A tree stands outside the city 20 meters north of the northern gate. One must walk 14 meters southward from the south gate and then turn west and walk 1775 meters before one can see the tree. What are the dimensions of the city?

A) 125 meters $\times$ 125 meters B) 284 meters $\times$ 284 meters C) 142 meters $\times$ 142 meters

D) 250 meters $\times$ 250 meters E) 267 meters $\times$ 267 meters

30) A tank that has the form of an inverted cone (see figure on the right) of radius 5 ft and height 15 ft contains 45 ft^3 of water. At time $t = 0$ a pump starts pumping water into the tank at a rate of 3 ft^3/min. Find the depth $h(t)$ in feet of the water in the tank t minutes later for $0 < t < 115$.



A) $h(t) = 3\left(\dfrac{45+3t}{\pi}\right)^{1/3}$ B) $h(t) = \dfrac{3}{\pi^3}(45+3t)^{1/3}$

C) $h(t) = 3\left(\dfrac{3t}{\pi}\right)^{1/3}$ D) $h(t) = \left(\dfrac{3(45+3t)}{\pi}\right)^{1/3}$

E) $h(t) = 3\left(\dfrac{45}{\pi}\right)^{1/3} + (3t)^{1/3}$

31) Find all values of a for which the equation $x^2 - 2ax + 2a^2 - 1 = 0$ has exactly one positive and one negative solution.

A) $-1 \le a \le 1$ B) $-1 < a < 1$ C) $-1 < a < -\dfrac{\sqrt{2}}{2}$ or $\dfrac{\sqrt{2}}{2} < a < 1$

D) $-\dfrac{\sqrt{2}}{2} \le a \le \dfrac{\sqrt{2}}{2}$ E) $-\dfrac{\sqrt{2}}{2} < a < \dfrac{\sqrt{2}}{2}$

32) * A patient has 300 milligrams of aspirin in her the bloodstream. If the amount of aspirin in her bloodstream decays exponentially, with one third being removed every 3 hours, find the amount $A(t)$ of aspirin in the bloodstream after t hours.

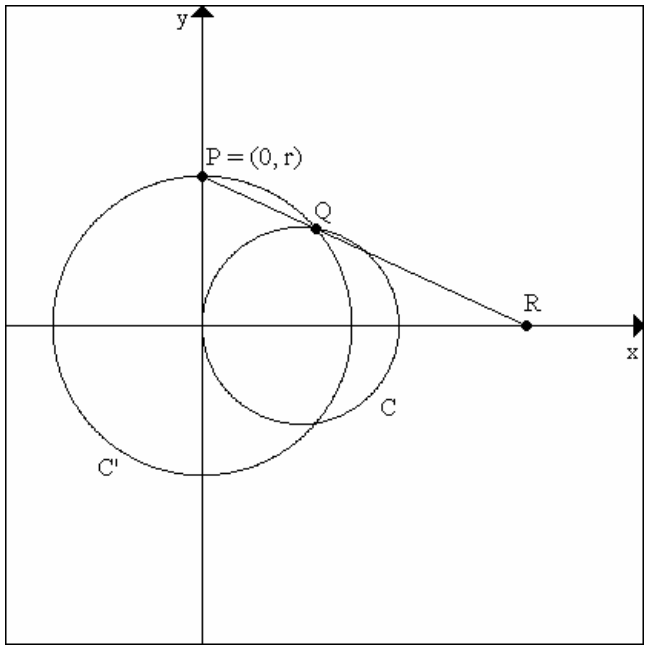
A) $A(t) = 300\left(\dfrac{1}{3}\right)^{\frac{t}{3}}$ B) $A(t) = 300\left(\dfrac{2}{3}\right)^{\frac{t}{3}}$ C) $A(t) = 300\left(\dfrac{1}{3}\right)^{3t}$

D) $A(t) = 300\left(\dfrac{2}{3}\right)^{3t}$ E) $A(t) = 300\left(\dfrac{1}{3}\right)^{\frac{2t}{3}}$

33) A pole of unknown length leans against a wall of height h so that its top is even with the top of the wall. If the bottom of the pole is moved 1 foot further from the wall, the pole will fall to the ground so that its top will touch the bottom of the wall. What is the length of the pole?

A) $\dfrac{h^2+1}{2}$ B) $\dfrac{h^2-1}{2}$ C) $\dfrac{1-h^2}{2}$ D) $2h^2+1$ E) $2\left(h^2+1\right)$

34) The figure shows a fixed circle C with equation $(x-1)^2+y^2=1$ and a circle C' with radius r, $0<r<2$ and center the origin. P is the point $(0,r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line segment $\overline{PQ}$ and the x-axis. Find the x coordinate of R.



A) $2-\sqrt{4-r^2}$ B) $2+\sqrt{4-r^2}$

C) $\dfrac{r^2}{\sqrt{4-r^2}+2}$ D) $\dfrac{-r^2}{\sqrt{4-r^2}+2}$

E) $\dfrac{r^2}{\sqrt{4-r^2}-2}$

35) If a,b are positive real numbers such that $a-b=2$, what is the smallest value of the constant L for which $\sqrt{x^2+ax}-\sqrt{x^2+bx}<L$ for all $x>0$?

A) $\dfrac{1}{2}$ B) $\dfrac{1}{\sqrt{2}}$ C) 1 D) 2 E) None of these

36) * Let $f(x)=(x+2)^2-1$ for $x\le -2$ and $g(x)=-2+\sqrt{x+1}$ for $x\ge -1$. Which of the following expressions gives $g(f(x))$ for $x\le -2$?

A) $-x-4$ B) x C) $-x$ D) $-2+\sqrt{(x+2)^2-2}$ E) $-2-|x+2|$

37) Which of the following is an algebraic expression for $\tan(\sin^{-1}(x))$, for $-1<x<1$?

A) $\dfrac{x}{\sqrt{x^2-1}}$ B) $\dfrac{x}{\sqrt{x^2+1}}$ C) $\dfrac{x}{1-x^2}$ D) $\dfrac{\sqrt{1-x^2}}{x}$ E) $\dfrac{x}{\sqrt{1-x^2}}$

38) * How many permutations of the letters ABCDEFG contain the word "BAD"?

A) 120 B) 600 C) 5040 D) 720 E) $\dfrac{7!}{4!3!}$

39) Two sets U, V are said to be equal if they have the same elements. Which of the following is equivalent to saying that U is not equal to V ?

I. There is an element in U which is not in V.
II. There is an element in V which is not in U.
III. There is an element in U which is not in V and there is an element in V which is not in U.
IV. There is an element in U which is not in V or there is an element in V which is not in U.
V. There is an element which is not in U and not in V.

A) I B) II C) III D) IV E) V

40) Let S be the set of all integers of the form $8t+7$ for some integer t and let T be the set of all integers of the form $4k+3$ for some integer k. Which of the following is true?

A) $T \subseteq S$ B) $S \subseteq T$ C) $S \cap T = \varnothing$

D) $S = T$ E) $S \cup T =$ "all odd integers"

41) A point is chosen at random on a line segment of length 1 dividing it into two segments. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

A) $\frac{2}{5}$ B) $\frac{1}{5}$ C) $\frac{2}{3}$ D) $\frac{3}{5}$ E) $\frac{1}{3}$

42) * A florist has in stock several dozens of each of the following: roses, carnations, and lilies. How many different bouquets of half a dozen flowers can be made?

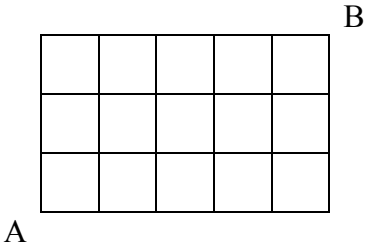
A) $\frac{8!}{2!6!}$ B) $\frac{9!}{3!6!}$ C) $\frac{12!}{6!6!}$ D) 56 E) 50

43) How many different committees of 5 members can be selected from a group of 14 men if David and Juan (two of the 14 men) refuse to serve together?

A) $\frac{12!}{4!8!}$ B) $2 \cdot \frac{12!}{4!8!}$ C) $\frac{12!}{4!8!} + \frac{12!}{5!7!}$

D) $2 \cdot \frac{12!}{4!8!} + \frac{12!}{5!7!}$ E) $\frac{14!}{5!9!}$

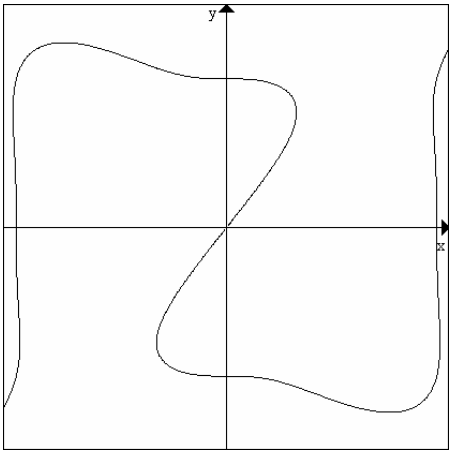
44) How many different routes are there from A to B in the following figure if we can travel only up or to the right?



A) $\frac{9!}{3!5!}$ B) $\frac{9!}{3!6!}$ C) $\frac{8!}{3!5!}$

D) 2^8 E) None of these

45) The figure on the right shows the graph of the equation $y^3\left(y^2-1\right)=x^3\left(x^2-2\right)$. Find the slope of the line tangent to the graph of the equation at the point $(0,0)$.



A) $\sqrt[3]{3}$ B) 2 C) $\sqrt{2}$

D) $\sqrt[3]{2}$ E) 1

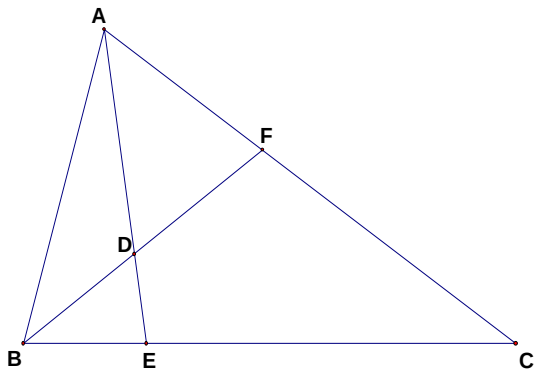
46) Let S be the set of all possible ordered pairs you can form using the numbers $1,2,3,4,5$. A nonempty subset R contained in S is called symmetric if every time a pair (a,b) is in R then the pair (b,a) is also in R. For instance, $\{(1,3),(3,1),(2,5),(5,2),(4,4)\}$ and $\{(2,2),(1,1)\}$ are examples of symmetric subsets. How many different symmetric subsets does S contain?

A) $2^{10}-1$ B) $2^{15}-1$ C) 2^5-1 D) 25 E) 15

47) Let $\triangle ABC$ be an equilateral triangle whose sides are 1 foot long. Five points are selected at random from the interior of the triangle. What is the probability that at least two of the selected points are at a distance less than $1/2$ foot?

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{4}$ E) $\frac{1}{5}$

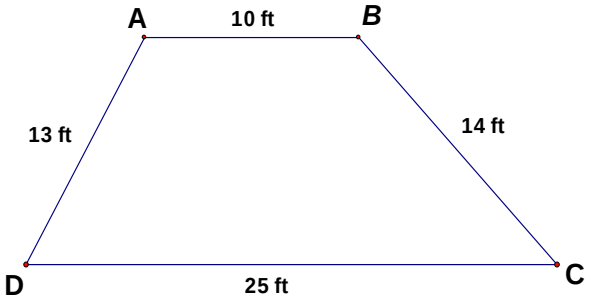
48) * The figure (not drawn to scale) shows a triangle $\triangle ABC$ with a point E on the side $\overline{BC}$ such that $\frac{BE}{EC}=\frac{1}{3}$ and a point D on $\overline{AE}$ such that $\frac{DE}{DA}=\frac{2}{5}$, find the ratio $\frac{FA}{FC}$.



A) $\frac{5}{8}$ B) $\frac{1}{2}$ C) $\frac{3}{8}$

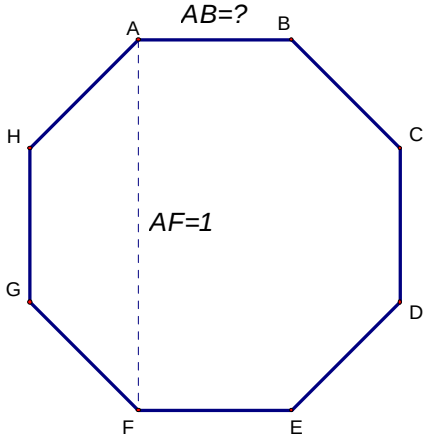
D) $\frac{3}{4}$ E) $\frac{1}{4}$

49) The figure (not drawn to scale) shows a trapezoid $ABCD$ with dimensions $AB=10$ ft, $BC=14$ ft, $DC=25$ ft, and $AD=13$ ft. Find the area of the trapezoid.



A) 196 ft^2 B) 196.44 ft^2 C) 206 ft^2 D) 186 ft^2 E) 187 ft^2

50) The figure (not drawn to scale) shows a regular octagon $ABCDEFGH$ with diagonal $AF=1$. Find AB.



A) $\sqrt{2}-1$ B) $\frac{1}{2}$ C) $\sqrt{3}-1$

D) $1-\frac{1}{\sqrt{2}}$ E) $3-2\sqrt{2}$