Forty Second Annual Columbus State University Invitational Mathematics Tournament

Sponsored by

The Columbus State University

Department of Mathematics

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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Introduction

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one best answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with No.2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (∗), in order of consideration are: 4, 10, 14, 16, 17, 21, 24, 28, 31, 36, 40, 41, 42, 43, 45, 46, 47 and 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$; $\angle A$ denotes the $\angle BAC$ or $\angle CAB$ in the triangle $\triangle ABC$. Pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

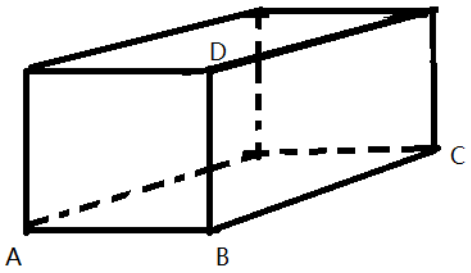
When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. If the straight lines $2y + 3x - 4 = 0$ and $3y - ax + 4 = 0$ are perpendicular to each other, what is the value of a?

   A) 5 *B*) 4 *C*) 3 *D*) 2 *E*) 1

2. In the cuboid represented below, we know $AB = BD = 1$ and $BC = 2$. What is the total surface area of the cuboid?



   A) 6 *B*) 2 *C*) 8 *D*) 9 *E*) 10

3. Let i be the complex number $\sqrt{-1}$. Find the value of i^{2016}.

   A) i *B*) $-i$ *C*) 1 *D*) -1 *E*) 0

4. *Find the minimum value of the function $f(x) = \dfrac{x+1}{x}$ for real numbers x, where $1 \leqslant x \leqslant 2016$.

   A) 1 *B*) $\dfrac{3}{2}$ *C*) $\dfrac{2017}{2016}$ *D*) 2 *E*) $\dfrac{2016}{2015}$

5. The average age of three girls is 7 years. If a boy joins the group, then the average age of these four children is 6 years. What is the boy's age?

   A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

6. For what value of k does the system of equations

$$\begin{cases} y = x^2 + 5 \\ y = 2x + k \end{cases}$$

   have a unique solution?

   A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

7. Find the radius of the circle $x^2 - 2x + y^2 - 4y - 4 = 0$.

$A)\,1$ $B)\,2$ $C)\,3$ $D)\,4$ $E)\,5$

8. Find the vertex of the parabola $y = x^2 - 2x + 3$.

$A)\,(1,2)$ $B)\,(1,-2)$ $C)\,(-1,2)$ $D)\,(0,3)$ $E)\,(2,3)$

9. What is the product of all solutions of the equation $x^4 - 2x^3 + 4x^2 + 6x + 3 = 0$?

$A)\,1$ $B)\,2$ $C)\,3$ $D)\,4$ $E)\,5$

10. *If the geometric mean of two positive real numbers a and b (with $a < b$) is equal to 15 and their arithmetic average is 39, what is $\sqrt{b/a}$?

$A)\,1$ $B)\,2$ $C)\,3$ $D)\,4$ $E)\,5$

11. If x is an integer, what is the minimum value of $(2x - 3)^2 + 2$?

$A)\,1$ $B)\,2$ $C)\,3$ $D)\,4$ $E)\,5$

12. There are 10 people in a room, 40% percent of whom are men. If no man enters or leaves the room, how many women must enter the room so that 20% of the total number of people in the room are men?

$A)\,10$ $B)\,9$ $C)\,3$ $D)\,7$ $E)\,6$

13. If $(1 - 2x)^{2016} = a_0 + a_1x + a_2x^2 + \cdots + a_{2016}x^{2016}$, find the value of $a_0 + a_1 + a_2 + \cdots + a_{2016}$.

$A)$ 5 $B)$ 4 $C)$ 3 $D)$ 1 $E)$ 2

14. *Assume that the sets $\{1, a+b, a\}$ and $\{0, \frac{b}{a}, b\}$ have the same elements for real numbers a and b. Find the value of $b - a$.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 0

15. If $(x - 3x^2)^3 = a_0 + a_1x + a_2x^2 + \cdots + a_5x^5 + a_6x^6$, find the value of $a_0 + a_1 - a_2$.

$A)$ 3 $B)$ 4 $C)$ 0 $D)$ 1 $E)$ 2

16. *How many real solutions does the equation $(x^2 - 3)^{x^2-2x} = 1$ have?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

17. *Each 3-digit positive integer is written on a card. All cards are placed in a box and one is extracted from the box. Find the probability that the sum of the digits on the card is 5.

A) $\frac{1}{900}$ *B)* $\frac{1}{300}$ *C)* $\frac{1}{150}$ *D)* $\frac{1}{90}$ *E)* $\frac{1}{60}$

18. Let x and y be real numbers such that $x^2 + 10x + 4y^2 - 20y + 50 = 0$. Find the value of $-2(x + y)$.

A) 1 *B)* 2 *C)* 3 *D)* 4 *E)* 5

19. If the real numbers x and y satisfy the system

$$\begin{cases} 3x + y &= k + 1 \\ x + 3y &= 3 \end{cases}$$

for the real number k with $2 < k < 4$, which one of the following is true about $x - y$?

A) $0 < x - y < 1/2$ *B)* $0 < x - y < 1$ *C)* $-3 < x - y < -1$

D) $-1 < x - y < 1$ *E)* $3 < x - y < 5$

20. Simplify the expression $\frac{4 \log \sqrt{b} \cdot \log_b a}{\log a}$ for $a > 1$ and $b > 1$.

A) 1 *B)* 2 *C)* 3 *D)* 4 *E)* 5

21. *If $f(x)$ is an odd function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$, what is the value of $f(2016)$?

A) 0 *B)* 1 *C)* 2 *D)* 3 *E)* 4

22. Find the number of distinct pairs (x, y) of positive integers which are solutions of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{7}$.

A) 1 *B)* 2 *C)* 3 *D)* 4 *E)* 5

23. Find all the values of a for which $B = \{x | x^2 - ax - 4 \leqslant 0\}$ contains the interval $[2, 4]$.

A) $[-1, 2)$ *B)* $[-1, 2]$ *C)* $[3, \infty)$ *D)* $[0, 3]$ *E)* $[2, \infty)$

24. *Consider the function $f(x) = \frac{5 - 4x + x^2}{2 - x}$ for $-\infty < x < 2$. Find the minimum value of $f(x)$.

A) 0 *B)* 1 *C)* 2 *D)* 3 *E)* 4

25. If x and y satisfy the following system of equations, find the value of xy.

$$\begin{cases} \log x - \log y = -2 \\ \log x + 2\log y = 1 \end{cases}$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

26. Find the value of a in $x^3 - 3x + a = 0$ if the sum of two of its solutions is 2.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

27. If x and b are real numbers such that $\log_{x^2} b + \log_{b^2} x = 1$, find the value of $\dfrac{x}{b}$.

A) 5 *B*) 4 *C*) 1 *D*) 2 *E*) 3

28. *If $3^{1-2x} = \pi$, find the value of 9^{1+x}.

A) $\dfrac{27}{\pi}$ *B*) $\dfrac{3}{\pi}$ *C*) $\dfrac{9}{\pi}$ *D*) $\dfrac{12}{\pi}$ *E*) $\dfrac{2}{\pi}$

29. Let $x = \sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}$. Find the value of $\left(x^3 + 2x^2 - 2x - 5\right)^{2016}$.

A) 5 *B*) 4 *C*) 3 *D*) 2 *E*) 1

30. Find the value of the product $\left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\left(1+\dfrac{1}{4}\right)\cdots\left(1+\dfrac{1}{2016}\right)$.

A) 2013 *B*) 2014 *C*) 2015 *D*) 2016 *E*) 2017

31. *How many triples (x, y, z) are solutions of the equation $x^2+y^2+z^2+\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} = 6$.

A) 2 *B*) 3 *C*) 4 *D*) 8 *E*) 6

32. Determine the number of real solutions of $x^2 - 3x - a^2 + 2 = 0$ for a real number a.

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

33. Find the interval on which the function $f(x) = \log_{\frac{1}{2}}(x^2 - 2x - 3)$ is increasing.

A) $(-\infty, +\infty)$ *B*) $(-\infty, -1)$ *C*) $(-1, 3)$ *D*) $(3, +\infty)$ *E*) $\left[3, +\infty\right)$

34. Find the value of $x+y$ such that x and y are real numbers which satisfy the following system of equations.

$$\begin{cases} (x-1)^3 + 2016(x-1) = -1 \\ (y-1)^3 + 2016(y-1) = 1 \end{cases}$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

35. Suppose $\log(a+b) = \log a + \log b$, for some real numbers a and b with $a > 1$ and $b > 1$. Find the value of $\log(a-1) + \log(b-1)$.

A) 2 *B*) 1 *C*) 0 *D*) 3 *E*) 5

36. *Suppose $\{a_1,\ a_2,\ a_3, \cdots\}$ is a geometric sequence of real numbers. The sum of the first n terms is denoted by S_n. If $S_{10} = 10$ and $S_{30} = 70$, calculate the value of $\dfrac{S_{40}}{50}$.

A) 1 *B*) 2 *C*) 3 *D*) 5 *E*) 4

37. Simplify the expression $\sqrt{x+1-2\sqrt{x}} + \sqrt{x+1+2\sqrt{x}}$ for $0 \leqslant x \leqslant 1$.

A) x^2 *B*) $2\sqrt{x}$ *C*) x *D*) 2 *E*) 1

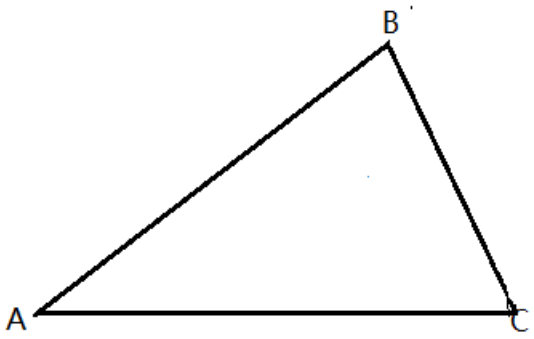
38. If $\tan(\theta) < 0$ and $\sin(\theta) = -\frac{1}{3}$, find $\cos(\theta)$.

A) $\sqrt{3}$ *B*) $\dfrac{\sqrt{3}}{3}$ *C*) $-\dfrac{\sqrt{3}}{3}$ *D*) $\dfrac{2\sqrt{2}}{3}$ *E*) $-\dfrac{2\sqrt{2}}{3}$

39. The function $f(x) = ax + b$ satisfies $f(f(f(1))) = 29$ and $f(f(f(0))) = 21$. Find the value of $a + b$.

A) 0 *B*) 1 *C*) 2 *D*) 4 *E*) 5

40. *In triangle $\triangle ABC$, if $\angle A = 60^\circ$, $AB = 2$, and $AC = 3$, find the length of the third side $\overline{BC}$.



A) $\sqrt{3}$ *B)* 2 *C)* $\sqrt{5}$ *D)* $\sqrt{6}$ *E)* $\sqrt{7}$

41. *If $\frac{5\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{2}$, find the value of $\sqrt{1-\sin 2\theta} - \sqrt{1+\sin 2\theta}$.

A) $2\sin\theta$ *B)* $-2\sin\theta$ *C)* $2\cos\theta$ *D)* $-2\cos\theta$ *E)* 0

42. *Find the minimum value of x^2+y^2 such that x and y satisfy the equation $(x+1)^2 + (y-1)^2 = 8$.

A) 2 *B)* 1 *C)* 0 *D)* 3 *E)* 4

43. *Find the range of $f(x) = 8(\sin^4 x + \cos^4 x - \sin x \cos x)$ for all real numbers x.

A) $(0,9)$ *B)* $[0,9)$ *C)* $(0,9]$ *D)* $[0,9]$ *E)* $[-8,8]$

44. Find $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{3x+8}-2}{x}$.

A) $\frac{5}{12}$ *B)* $\frac{7}{12}$ *C)* $\frac{1}{12}$ *D)* $\frac{1}{2}$ *E)* $\frac{1}{4}$

45. *The integers x and y satisfy the following system

$$\begin{cases} 5^x - \log_2(y+3) &= 3^y \\ 5^y - \log_2(x+3) &= 3^x \end{cases}$$

Find the value of $x+y$.

A) -2 *B)* -1 *C)* 0 *D)* 2 *E)* 3

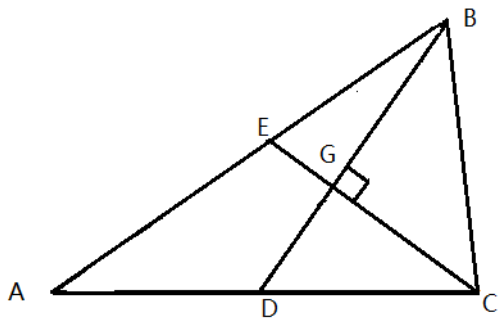
46. *Let $f(x)$ be a real-valued function defined on $(-\infty,\infty)$ such that $f(f(x)) = 2^x - 1$. Find the value of $f(0)+f(1)$.

A) 1 *B)* 2 *C)* 0 *D)* -2 *E)* -1

47. *Find the number of ordered pairs of real numbers (a,b) such that $(a+bi)^{2016} = a-bi$, where $i = \sqrt{-1}$.

A) 2014 *B)* 2015 *C)* 2016 *D)* 2017 *E)* 2018

48. In the triangle $\triangle ABC$, $AB = 2\sqrt{13}$, $AC = \sqrt{73}$, E and D are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. $\overline{BD}$ is perpendicular to $\overline{CE}$. Find the length of $\overline{BC}$.



A) 3 *B*) 4 *C*) 5 *D*) 6 *E*) 7

49. If $0 \leqslant \theta \leqslant \pi$ and $x + \dfrac{1}{x} = 2\cos\theta$, find the value of $x^3 + \dfrac{1}{x^3}$.

A) $2\cos\theta$ *B*) $2\cos 3\theta$ *C*) $3\cos 3\theta$ *D*) $8\cos^3\theta$ *E*) $3\cos^3\theta$

50. *Find the number of triples (x, y, z) of rational numbers which are solutions of the following system.

$$\begin{cases} x + y + z &= 0 \\ xyz + z &= 0 \\ xy + yz + zx + y &= 0 \end{cases}$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5