Forty Third Annual Columbus State University Invitational Mathematics Tournament

Sponsored by

The Columbus State University

Department of Mathematics

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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Introduction

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one best answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No.2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 6, 24, 26, 29, 32, 34, 37, 39, 40, 41, 42, 43, 44, 46, 47, 49, and 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$, $\angle A$ denotes the $\angle BAC$ or $\angle CAB$ in the triangle $\triangle ABC$. Pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. If lines $2y + 3x - 4 = 0$ and $3y - ax - 4 = 0$ intersect at a point (x_1, y_1), what is the value of $\frac{y_1}{x_1} - a$?

   A) 5 *B*) 4 *C*) 2 *D*) 3 *E*) 1

2. A fair coin is flipped three times. What is the probability that at least one head will be thrown ?

   A) 0.875 *B*) 0.125 *C*) 0.25 *D*) 0.5 *E*) 0.75

3. If $i^2 = -1$, find the value of i^{2017}.

   A) i *B*) $-i$ *C*) 1 *D*) -1 *E*) 0

4. If $(0.67)^H = 0.5$, what is the value of $2^4(0.67)^{3H}$?

   A) 3 *B*) 2 *C*) 3 *D*) 4 *E*) 5

5. How many real numbers are solutions of the following equation?

$$x - \frac{3}{(x-2)^4} = 2 - \frac{3}{(x-2)^4}$$

   A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

6. * How many different nonnegative real numbers satisfy the following equation?

$$(x^2 + 4x - 2)^2 = (5x^2 + 2)^2$$

   A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

7. How many real numbers x satisfy equation $(x^2 - x + 1)^{(x-1)} = 1$?

   A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

8. A certain two digit number is equal to twice the sum of its digits. What is the product of its digits ?

   A) 4 *B*) 6 *C*) 7 *D*) 5 *E*) 8

9. Find the area of the region bounded by the $x-$axis, $y-$axis, and $\frac{x}{3} - \frac{y}{4} = 1$.

   A) 10 *B*) 9 *C*) 3 *D*) 7 *E*) 6

10. Assume a and b are constants such that the following equality

$$\frac{a}{x-1}+\frac{b}{x-2}=\frac{2017}{(x-1)(x-2)}$$

is true for any real number x except 1 and 2. What is the value of $a+b$?

A) 4 *B*) 3 *C*) 2 *D*) 1 *E*) 0

11. Suppose that a and b are real numbers such that $1+\sqrt{3}i$ is a root of the quadratic equation $\frac{x^2}{2}+ax+b=0$. Find the value of b.

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

12. If $9^{1-\frac{3}{2}x}=\pi$, find the value of $6\cdot\pi(27)^{x-1}$.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

13. Let x, y be real numbers such that

$$\begin{cases} xy-x=25, \\ xy+y=36. \end{cases}$$

What is the value of $y-x$?

A) 1 *B*) 2 *C*) 0 *D*) -2 *E*) -1

14. What is the product of the roots of the following equation?

$$(x+1)(x-2)+(x+2)(x-3)+(x+3)(x-1)+(x+3)(x+5)=0.$$

A) 2 *B*) 1 *C*) 0 *D*) 3 *E*) 5

15. The quadratic function $f(x)=sx^2+tx+r$ satisfies the equation $f(x)-f(x-1)=4x+1$ for all x. What is the value of $t-s$?

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

16. If $\frac{1}{a}+\frac{1}{b}=\frac{2017}{a+b}$, find the value of $\frac{b}{a}+\frac{a}{b}$.

A) 2014 *B*) 2015 *C*) 2016 *D*) 2017 *E*) 2018

17. Find the length of the **minor** axis(the shortest diameter of an ellipse) of the ellipse $x^2+5y^2=5$.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

18. Given numbers k, l and m such that

$$x^3 + kx^2 - x - 2 = (lx - 1)(x^2 + mx + 2)$$

is an identity in x, what is the value of $m - k$?

$A)\, 3$ $B)\, 4$ $C)\, 0$ $D)\, 1$ $E)\, 2$

19. Let $\frac{a+2}{a+1} = pa + q$ and $2a^2 - 3a - 8 = 0$. Find the value of $p + q$.

$A)\, 0$ $B)\, 1$ $C)\, 2$ $D)\, 3$ $E)\, 4$

20. Let F be a function from positive real numbers to positive real numbers such that $F(2) = a$. If $[F(xy)]^2 = x[F(y)]^2$ for all positive real numbers x and y, find the value of $F(50)$.

$A)\, a$ $B)\, 2a$ $C)\, 3a$ $D)\, 4a$ $E)\, 5a$

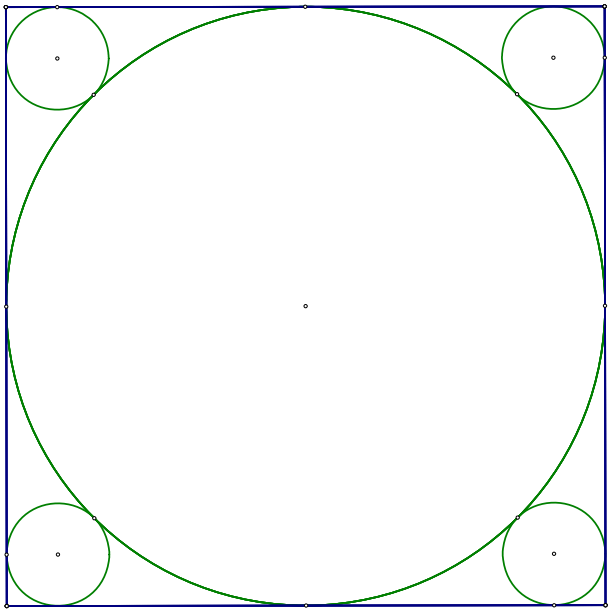
21. If $\log_a x = 4$, what is the value of $\log_{\frac{1}{a}} x$?

$A)\, -\frac{1}{4}$ $B)\, -4$ $C)\, 1$ $D)\, 4$ $E)\, \frac{1}{4}$

22. A bag contains 10 chips that are either red or blue. It is known that the probability of selecting (without replacement) a red chip and then a blue chip (in that order) is $\frac{4}{15}$, and that the probability of selecting two blue chips is $\frac{1}{3}$. How many red chips are in the bag?

$A)\, 3$ $B)\, 4$ $C)\, 5$ $D)\, 6$ $E)\, 2$

23. In the accompanying figure we have a square and a large circle is inscribed in the square. The smaller circles are tangent to the large circle and to the sides of the square. If the sides of the square are equal to 2 cm, then the radii of the smaller circles are equal to $a - b\sqrt{2}$ cm, for some natural numbers a and b. What is $a - b$?



A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 0

24. * What is the coefficient of x^4y^2 in the expansion of $(x+2y)^6$?

A) 20 *B*) 30 *C*) 40 *D*) 50 *E*) 60

25. The equation $r = 6\cos\theta$ is a circle written in polar form. What is the radius of the circle ?

A) 6 *B*) 5 *C*) 4 *D*) 3 *E*) 2

26. * Compute the value of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{300}$, where $i^2=-1$.

A) -1 *B*) 1 *C*) 2 *D*) $\sqrt{3}$ *E*) $-\sqrt{3}$

27. Let a and b be complex numbers such that $\frac{a}{b}+\frac{b}{a}=\sqrt{3}$. What is the value of a^6+b^6 ?

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 0

28. If $\log_x 64 = 3$, find the value of $\log_2\left(x^3\right)$.

A) 5 *B*) 4 *C*) 6 *D*) 2 *E*) 3

29. * For each real number m the parabola $y=(m^2+2)x^2+(m-1)^2x-2m+2$ passes through the same point (a,b). What is the value of $(a+b)^2$?

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

30. Find the product of the solutions of the equation

$$\log_3 x - 4\log_x 3 - 3 = 0.$$

A) 12 *B*) 9 *C*) 16 *D*) 27 *E*) 36

31. Find the interval on which the function $f(x)=\log_{\frac{1}{2}}(-x^2-2x+3)$ is increasing.

A) $(-\infty,+\infty)$ *B*) $(-\infty,-3)$ *C*) $\left(-3,-1\right]$ *D*) $\left(-1,1\right)$ *E*) $\left[-1,1\right]$

32. * Find the number of nonnegative integers n such that $\frac{n^2}{n+6}$ is an integer.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

33. What is the value of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{2017}\right)$?

A) $\frac{2017}{2}$ *B)* 2017 *C)* $\frac{2018}{3}$ *D)* 1009 *E)* 1010

34. * Solve the equation $x=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$.

A) 1 *B)* 2 *C)* 0 *D)* -2 *E)* -1

35. A point is chosen at random on a line segment of length 1 dividing it into two segments. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

A) $\frac{2}{5}$ *B)* $\frac{1}{5}$ *C)* $\frac{3}{5}$ *D)* $\frac{1}{3}$ *E)* $\frac{2}{3}$

36. Let $S=\cos 1^\circ+\cos 3^\circ+\cos 5^\circ+\cdots+\cos 29^\circ$. What is the value of $S\sin 1^\circ$?

A) $\frac{5}{4}$ *B)* 1 *C)* $\frac{3}{4}$ *D)* $\frac{1}{4}$ *E)* $\frac{1}{2}$

37. * What is the 2017^{th} term of the sequence

1, 2, 2, 3, 3, 3, 4, 4, 4, 4,...?

A) 62 *B)* 63 *C)* 64 *D)* 65 *E)* 66

38. Let $\sin\alpha\cos\alpha=\frac{1}{8}$ and $\pi/4<\alpha<\pi/2$. Find the value of $\cos\alpha-\sin\alpha$.

A) $\frac{\sqrt{3}}{2}$ *B)* $-\frac{\sqrt{3}}{2}$ *C)* $\frac{3}{4}$ *D)* $-\frac{3}{4}$ *E)* $\frac{1}{2}$

39. * Let θ be a real number such that $\sin\theta+\csc\theta=2$. Find the value of $\sin^3\theta+\csc^3\theta$.

A) $2\sin\theta$ *B)* $-2\sin\theta$ *C)* $2\cos\theta$ *D)* $-2\cos\theta$ *E)* 2

40. * Let $P(x,y)$ be any point on the circle $r=2\cos\theta-2\sin\theta$. Find the minimum value of x^2+y^2.

A) 2 *B)* 1 *C)* 0 *D)* 3 *E)* 4

41. * Three balls are randomly drawn, without replacement, from a bowl containing 6 yellow and 5 blue balls. Find the probability that one of the balls is yellow and the other two are blue.

A) $\frac{4}{11}$ *B)* $\frac{3}{11}$ *C)* $\frac{5}{11}$ *D)* $\frac{6}{11}$ *E)* $\frac{7}{11}$

42. * Let a, b, c, and d be positive real numbers such that $ab = cd = 1$. Find the smallest possible value of the expression

$$S = \frac{(a+1)(b+1)(c+3)(d+3)}{16}.$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

43. * Suppose that $x_1, x_2, \cdots, x_{2017}$ are the 2017 roots of the polynomial

$$P(x) = x^{2017} - 2017^{2016}x - 2.$$

Find the average of $x_1^{2017}, x_2^{2017}, \cdots, x_{2017}^{2017}$.

A) -2 *B*) -1 *C*) 0 *D*) 2 *E*) 3

44. * Let a, b, c, and d be positive real numbers such that $\dfrac{a+b}{c+d} = \dfrac{2a+b}{2c+d} = \dfrac{3a+b}{3c+d} = 3$. What is the value of $\dfrac{2017a+b}{2017c+d}$?

A) 2014 *B*) 2015 *C*) 1 *D*) 2 *E*) 3

45. If $A = 3^{2017} - 3^{2016} + 3^{2015} - 3^{2014}$, what are the tens and ones digits of A?

A) 10 *B*) 20 *C*) 40 *D*) 60 *E*) 80

46. * Let a, b, and c be positive numbers such that $a + b + c = 9$ and

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{1}{3}.$$

What is the value of $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$?

A) 1 *B*) 2 *C*) 0 *D*) 4 *E*) 3

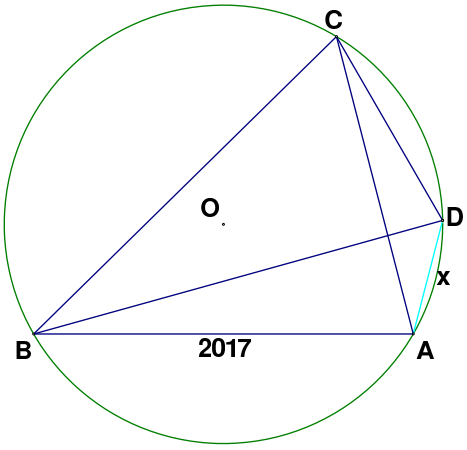
47. * If positive integers x and y satisfy the equation $2017 + 2016i = (48 + 16i)^2 + (y + xi)^2$, what is $x - y$?

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

48. Two points are chosen at random on a square (uniform distribution on each side). The probability that the distance between them is more than or equal to the side of the square is equal to $\frac{m-\pi}{n}$ for two natural numbers m and n. What is $n - m$?

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

49. * In the accompanying figure ABCD is a cyclic quadrilateral (inscribed in a circle) in which $AB = 2017$ cm, $BD = BC = 2255$ cm, and $AC = 1632$ cm. The length of the chord $x = AD$ (in centimeters) is a number of three digits (in base 10). What is the difference between the first and the second digit (the hundreds digit minus the tens digit) of x?.



A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

50. * The number 1729 is the Hardy-Ramanujan number, i.e., the smallest integer which is the sum of two cubes in two different ways ($1729 = 1^3 + 12^3 = 9^3 + 10^3$). This allows us to write 2017 as a sum of five different positive cubes (of natural numbers) in two different ways:

$$2017 = 1^3 + 12^3 + a^3 + b^3 + c^3 = 9^3 + 10^3 + a^3 + b^3 + c^3.$$

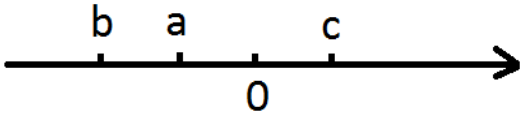
If $a < b < c$, find $(a - b + c)/2$.

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

Math Tournament 2017 - Ciphering

Round 1

1. Let a, b and c be points equally spaced in the number line as shown below.



Simplify the expression

$$\sqrt{a^2} - |a+b| + \sqrt{(c-a)^2} + |b+c|.$$

2. If $a = \sqrt{2}+1$, what is the value of $1 + \dfrac{1}{2+\frac{1}{a}}$?

3. One third of the number 27^{81} can be written in the form 9^A where A is a three digit number. Find the value of A.

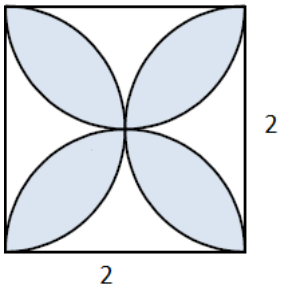
4. If $2x^2 + y^2 - 2xy - 4x + 4 = 0$, what is the value of $x + y$?

5. Let a and b be real numbers such that $a \neq b$, $a^2 + 3a = 2$, and $b^2 + 3b = 2$. Find the value of $(1+a)(1+b)$.

6. Evaluate $\log_2 2\sqrt{4\sqrt{2}} - \frac{1}{4}$.

7. The sum of two prime numbers is 99. What is the product of these two prime numbers?

8. The figure shows a square and four semicircles generated with each side of the square as a diameter. If the side length of the square is 2, find the area of the shaded region.



Round 2

1. How many digits are in the base-ten numeral $16^{11} \cdot 5^{40}$?

2. Amber, Ben, and Cathy shared a pizza. Amber ate $\frac{1}{5}$ of the pizza, Ben ate one-half as much as Cathy did. Find how much of the pizza Ben ate.

3. Let a and b be real numbers such that $a + b = 1$ and $a^2 + b^2 = 4$. What is the value of $\frac{b^2}{2+a} + \frac{a^2}{2+b}$?

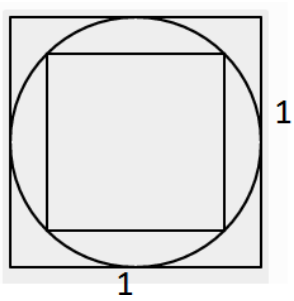
4. Let a be a real number such that $a > 0$ and $|(a + 2i)(1 + i)| = 4$. Find the value of a.

5. In a barn with chickens and rabbits, there are 7 heads and 20 legs. How many chickens are there? (A chicken has 2 legs and a rabbit has 4 legs.)

6. Let $f(x) = (2 - \sin\sqrt{x})^2$. What is the maximum value of $f(x)$?

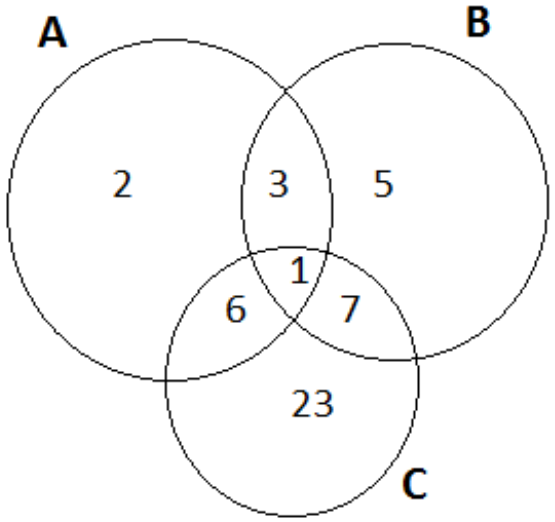
7. Let $M = \{1, 2, 3\}$ and $N = \{2a - 1 | a \in M\}$. Find $M \bigcap N$.

8. If the area of the outer square is 1, what is the area of the small square inside?



Backup Questions

1. Find the elements of set $B' \cap C$ in the following Venn diagram (where B' denotes the complement of set B).



2. If $x = -1$ is a triple root of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + 2$, what is the value of a ?

3. Simplify $\dfrac{\sqrt{5}+2}{\sqrt{5}-2} - 4\sqrt{5}$.