Thirty-fourth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
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The Mathematics Department at Columbus State University welcomes you to our campus and to this year's tournament. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200. Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 21, 24, 31, 34, 36, 40, 41, and 42.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.** When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) If $x+y=0$ and $x\neq 0$, what is the value of $\dfrac{(2y)^{2008}}{2x}\div(x-y)^{2007}$?

A) -1 B) 1 C) y D) x E) 2

2) If a point (x, y), not on the axes, satisfies both of the inequalities $y<3x$ and $y<-2-x$, in which quadrants could the point be located?

A) I, II B) II, III C) I, III D) III, IV E) I, IV

3) What are the last two digits of 5^{2008}.

A) 75 B) 65 C) 35 D) 25 E) 15

4) Little Billy's score on this year Columbus State Invitation Mathematics Tournament test is 410. (See the rule on the first page.) His score would be 32 if he had been given +2 points for each correct answer, -1 point for each incorrect answer, 0 for each left blank, and an initial score of 0. How many did he leave blank?

A) 10 B) 15 C) 25 D) 26 E) 30

5) It takes 40 minutes for Moe to paint a room alone and it takes 60 minutes for Joe to paint the same room alone. If they decide to paint the room together, how long will it take them to paint one-third of the room?

A) 20 min B) 8 min C) 10 min D) 12 min E) 24 min

6) A tank contains 10 gallons of a 90% salt solution. How many gallons of a 50% salt solution must be added to the tank to yield a 70% salt solution?

A) 6 gal B) 7 gal C) 8.5 gal D) 9 gal E) 10 gal

7) Dick and Jane have a total of $480. Dick takes one-third of Jane's money. Then Dick takes one-fourth of the money that Jane has left. Jane and Dick now have equal amounts of money. How much money did Jane have originally?

A) $480 B) $350 C) $320 D) $300 E) $280

8) Exactly $3/4$ of the boys and exactly 4/5 of the girls in a class passed an algebra test. If an equal number of boys and girls passed the test, what fraction of the entire class passed the test?

A) 12/15 B) 4/7 C) 16/ 21 D) 18/25 E) 24/31

9) Company A will rent a car for $35 per day plus $0.10 per mile, whereas company B charges $30 per day plus $0.12 per mile. John needs a car for 5 days. For what range of mileage will John save money by renting a car from company B?

A) More than 1000 miles B) No more than 1200 miles

C) More than 1210 miles D) More than 1250 miles

E) Less than 1250 miles

10) Two vertical poles, 16ft and 24ft high, are erected 13 feet apart on level ground. Straight wires are attached from the top of each pole to the bottom of the other pole. What is the vertical distance from the ground to the intersection of the wires?

A) 5.6 ft B) 7ft C) 8ft D) 8.6ft E) 9.6ft

11) What is the coefficient of x^6y^5 in the expansion of $(2x-3y)^{11}$?

A) -1077536 B) 1077536 C) -7185024 D) 7185024 E) -15552

12) A three-digit number equals 19 times the sum of its digits. If the digits are reversed, the resulting number is greater than the given number by 297. The tens digit exceeds the units digits by 3. Find the number.

A) 184 B) 285 C) 386 D) 487 E) 588

13) A certain rectangle has an area of 120 square inches. Increasing its width by 4 inches and decreasing its length by 3 inches increases its area by 24 square inches. Find the perimeter of the original rectangle.

A) 44 in B) 46 in C) 52 in D) 68 in E) 86 in

14) How many ones are there if 2008 (base 10) is written in base 2?

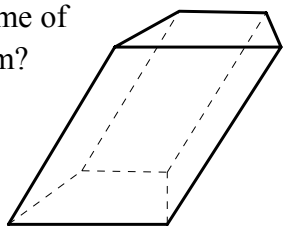
A) 5 B) 6 C) 7 D) 8 E) 9

15) Find the sum of the digits of the only real solution to the equation

$$\frac{7}{x-1}-\frac{2}{\sqrt{x-1}}+\frac{1}{7}=0.$$

A) 1 B) 2 C) 3 D) 4 E) 5

16) A 1:40 scale model of a prism is shown on the right. If the volume of the scaled model is $5\ \text{cm}^3$, what is the volume of the actual prism?



A) $200\ \text{cm}^3$ B) $8000\ \text{cm}^3$ C) $0.2\ \text{m}^3$

D) $0.32\ \text{m}^3$ E) $0.032\ \text{cm}^3$

17) Find the sum of all real and complex solutions of the equation $x^3 = 27$.

A) 0 B) 3 C) -6 D) $3\sqrt{3}i$ E) $-3\sqrt{3}i$

18) Solve the equation $2^{16^x} = 16^{2^x}$.

A) 1/2 B) 1 C) 2 D) 3 E) 2/3

19) Find the real solution of the equation $2^x - 2^{-x} = \frac{8}{3}$.

A) 3/2 B) $\log_2 3$ C) 0.7 D) 0.8 E) $\log_3 2$

20) If $\cos x = 2\sin x$, then what is the value of $\sin x \cos x$?

A) 2/3 B) 2 C) 3 D) 2/5 E) 4/5

21) * Define a sequence by $b_1 = 3$ and $b_{n+1} = \frac{1+b_n}{1-b_n}$ for integers $n \geq 1$. What is the value of b_{2008}?

A) 3 B) 1 C) $\frac{1}{2}$ D) -2 E) $-\frac{1}{3}$

22) A game is played by selecting a four-digit number from 0000 to 9999, and the winning prize is \$5000. Suppose you pay \$1 to play the game. What is the expected value of your gain or loss in dollars?

A) 2500 B) 1 C) -0.2 D) -0.4 E) -0.5

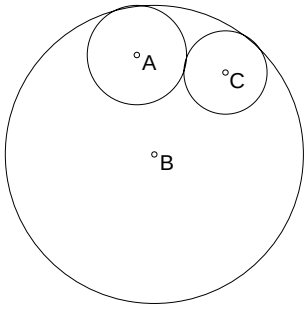
23) Suppose there are five problems on a test with 5 possible responses for each problem. If you guess each answer, what is the probability that you get more than 3 correct answers?

A) 0.0067 B) 0.005 C) 0.075 D) 0.05 E) 0.5

24) * How many zeros are there at the end of 2008!?

A) 0 B) 450 C) 500 D) 2008 E) 550

25) The three circles in the figure centered at A, B, and C are tangent to one another and have radii 7, 21 and 6, respectively. Find the area of triangle ABC.



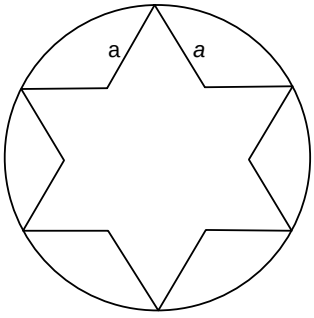
A) 54 B) 64 C) 74

D) 84 E) 94

26) How many distinct permutations are there of the letters in the word "MATHEMATICS", using all the letters?

A) 4989600 B) 6652800 C) 39916800 D) 332640 E) 55440

27) A regular six-pointed star is inscribed in a circle. Let a be the edge length. Express the area of the star in terms of a.



A) $3\sqrt{2}a^2$ B) $\sqrt{3}a^2$ C) $3\sqrt{3}a^2$

D) $6\sqrt{2}a^2$ E) $\sqrt{6}a^2$

28) If $a > 0,$ what is the value of $\left[\log_a\left(\frac{\log_a a^3}{3a}\right)\right]^{100}$?

A) 0 B) 1 C) a D) $\log_a 3$ E) 100

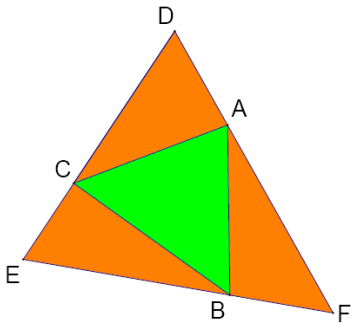
29) For $-1 \le x \le 1$ and $x \ne 0$, what is the formula for $\cot(\sin^{-1} x)$?

A) $\dfrac{\sqrt{1-x^2}}{x}$ B) $\dfrac{1}{x}$ C) $\sqrt{1-x^2}$ D) x E) $\dfrac{x}{\sqrt{1-x^2}}$

30) Let $Y = 104^4 + 4(104^3) + 6(104^2) + 4(104) + 1$. How many positive integer factors does Y have?

A) 216 B) 4 C) 5 D) 6 E) 125

31) * In the triangle ΔDEF, the points $A, B,$ and C are located on the sides of the triangle (see figure), such that $DC = 2EC$, $EB = 2BF$, and $FA = 2AD$. Find the ratio $(\text{Area } \Delta DEF)/(\text{Area } \Delta ABC)$.



A) 3/2 B) 4/3 C) 2

D) 3 E) 7/3

32) Given a circle of radius 1, what is the probability that the length of the chord between any two randomly chosen points on the circle will be greater than or equal to 1?

A) 2/3 B) 1/3 C) 1/4 D) 3/4 E) 1/6

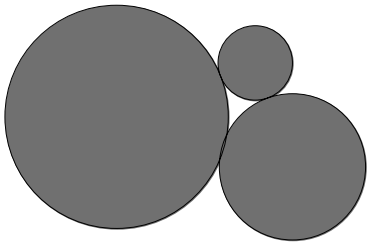
33) Five distinct integers are picked at random from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. What is the probability that, among those selected, the third smallest is 3?

A) $\dfrac{5}{28}$ B) $\dfrac{5}{45}$ C) $\dfrac{5}{84}$ D) $\dfrac{1}{168}$ E) $\dfrac{1}{38}$

34) * How many distinct permutations are there of 3 letters selected from the letters in the word "MATHEMATICS"?

A) 381 B) 336 C) 399 D) 165 E) 990

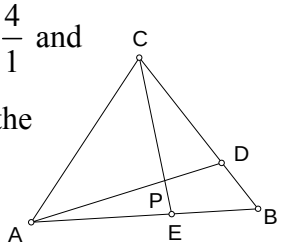
35) Three mutually tangent circles have radii 1, 2, and 3 inches, respectively. Find the area of the white region between the circles.



A) $1\ \text{in}^2$ B) $2\ \text{in}^2$ C) $0.34\ \text{in}^2$

D) $0.46\ \text{in}^2$ E) $0.75\ \text{in}^2$

36) * In triangle ΔABC lines CE and AD are drawn so that $\frac{CD}{DB}=\frac{4}{1}$ and $\frac{AE}{EB}=\frac{4}{3}$. Let P be the intersection point of CE and AD. Find the ratio $\frac{CP}{PE}$.



A) 3 B) 4 C) 5 D) 6 E) 7

37) Suppose x is a complex number satisfying the equation $x+\frac{1}{x}=2\cos\frac{\pi}{k}$ for some natural number k. What is the value of x^k+x^{-k}?

A) 0 B) 2 C) -2 D) k E) $-k$

38) If $\sin x+\sin y=\frac{1}{\sqrt{2}}$ and $\cos x+\cos y=\sqrt{2}$, what is the value of $\cos(x-y)$?

A) 1 B) 1/4 C) $-1/2$ D) 2/3 E) $-2/3$

39) The distance to three corners of square $ABCD$ from a point X in its interior is $XA=10$, $XB=20$, and $XC=30$. Find the distance XD.

A) 8 B) 25 C) $2\sqrt{10}$ D) $10\sqrt{3}$ E) $10\sqrt{6}$

40) * Every natural number which is not a power of 2 can be written as a sum of two or more consecutive integers. There is only one representation of 2008 in the form $2008=a+(a+1)+\cdots+b$ when $a>0$. Find $a+b$.

A) 16 B) 151 C) 251 D) 44 E) 45

41) * If the sum of two solutions of the equation $x^4 - x^3 - 2x^2 + 3x - 3 = 0$ is 0, find the other two solutions.

A) $\frac{1 \pm i\sqrt{3}}{2}$ B) $\frac{1 \pm i\sqrt{2}}{2}$ C) $\pm\sqrt{3}$ D) $\pm\sqrt{2}$ E) $1 \pm i$

42) * Let $f(x)$ be a polynomial of degree 2007 satisfying

$$f(k) = \frac{1}{k+1} \quad \text{for} \quad k = 1, 2, \ldots, 2008.$$

What is the value of $f(-2)$?

A) 1004 B) 1490 C) 0 D) 1 E) -1

43) If $aabb$ is the representation of 2008 in base c and $a, b > 0$, what is $a - 2b + c$?

A) 0 B) 1 C) 2 D) 3 E) 4

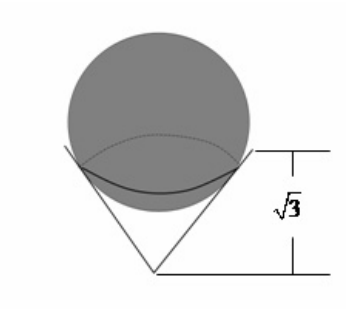
44) For a real number $a > 0$, how many real solutions satisfy the equation $\sqrt{a + \sqrt{a + x}} = x$?

A) 0 B) 1 C) 2 D) 3 E) 4

45) Find the sum of the digits of the greatest integer n with the property that $n + 2007$ divides $n^2 + 2008$.

A) 12 B) 17 C) 18 D) 19 E) 212

46) A 2 inch diameter spherical scoop of ice cream rests on cone that has a 2 inch diameter opening and $\sqrt{3}$ inch height. Find the diameter of the intersection of the sphere and the cone.



A) 2 in B) 1.5 in C) $\sqrt{2}$ in
D) $\sqrt{3}$ in E) $\frac{\sqrt{3}}{2}$ in

47) Find the sum $a + 2a^2 + 3a^3 + 4a^4 + \cdots$ if $0 < a < 1$.

A) $\frac{a}{1+a}$ B) $\frac{a}{1-a}$ C) $\frac{a}{(1-a)^2}$ D) $\frac{a}{1-a^2}$ E) $\frac{a^2}{1-a^2}$

48) Find the value of a in the parabola $x = 1 + ay^2$ so that the line $y = x + 1$ is a tangent line to the parabola.

A) 1/4 B) 1/8 C) 7/64 D) 2/17 E) 9/16

49) Two cars are traveling at the same constant speed toward each other on a straight road. A plane flying 350 miles per hour passes over the second car two hours after passing over the first car. The plane continues to fly in the same direction and is 2400 miles from the cars when they pass each other. Find the speed of the cars.

A) 50 mph B) 55 mph C) 60 mph D) 65 mph E) 70 mph

50) Evaluate the nonterminating continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

A) 3 B) 2 C) $\sqrt{2}$ D) $\frac{1+\sqrt{5}}{2}$ E) $\frac{1-\sqrt{5}}{2}$