**Forty-Fourth Annual Columbus State Invitational Mathematics Tournament**

Sponsored by
The Columbus State University
Department of Mathematics
March $3^{rd}$, 2018

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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

## Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, $-3$ for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 4, 9, 11, 13, 14, 15, 16, 21, 26, 27, 30, 33, 34, 35, 36, 37, 39, 42, 45, 47, 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point $A$ to point $B$ and $AB$ will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. How many positive integer divisors does 2018 have?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

2. How many of the first 2018 positive integers are perfect squares?

(A) 44 (B) 45 (C) 43 (D) 42 (E) 41

3. For integers $a$ and $b$, we define $a \otimes b$ by the formula

$$a \otimes b = \mathrm{gcf}(a, b) \cdot \mathrm{lcm}(a, b),$$

where "gcf" and "lcm" are the greatest common factor and least common multiple of $a$ and $b$, respectively. Find the value of $18 \otimes 75$.

(A) 675 (B) 270 (C) 1080 (D) 450 (E) 1350

4. $*$ When a certain solid substance melts, its volume increases by $\frac{1}{7}$. By how much does its volume decrease when it solidifies again?

(A) $\frac{1}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{10}$ (E) $\frac{1}{11}$

5. The equation

$$2017 + 2018i = (3 - 2i)(x + yi)$$

has a solution in positive integers $x$ and $y$. What is $2y - 10x$?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

6. If 45 is the sum of $n$ consecutive positive integers, what is the largest possible value on $n$?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

7. How many triples $(x, y, z)$, of integer numbers, $x > y > z > 1$, satisfy the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

8. How many 2-digit positive integers can be represented as the sum of different powers of 2?

(A) 30 (B) 45 (C) 60 (D) 75 (E) 90

9. $*$ For what value of the real number $a$ does the system of equations

$$\begin{cases} x^2 + y^2 = z \\ x + y + z = a \end{cases}$$

have a unique solution in the set of real numbers?

(A) $\frac{1}{4}$ (B) $\frac{1}{5}$ (C) $\frac{1}{2}$ (D) $-\frac{1}{3}$ (E) $-\frac{1}{2}$

10. Find the sum of the real numbers $x$ and $y$ which satisfy the equation

$$\sqrt{4x^2 - 12x + 25} + \sqrt{4x^2 + 12xy + 9y^2 + 64} = 12.$$

(A) $\frac{7}{2}$ (B) $\frac{3}{2}$ (C) $\frac{9}{2}$ (D) $\frac{1}{2}$ (E) $\frac{5}{2}$

11. $*$ The positive real numbers $x$ and $y$ satisfy the equation

$$x^2 - 2xy - 3y^2 = 0.$$

What is the value of $\frac{x + 2y}{x - y}$?

(A) 0 (B) 1 (C) $\frac{5}{2}$ (D) 3 (E) 4

12. Three distinct prime numbers $p$, $q$ and $r$ are chosen in such a way the number

$$p^4 + q^4 + r^4 - 3^5$$

is also a prime. What is the minimum possible value of $|pq - r|$?

(A) 2 (B) 1 (C) 4 (D) 0 (E) 3

13. $*$ What is the units digit of $2018^{2018}$?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

14. $*$ There is only one 4-digit integer $n$ for which $\sqrt{3\sqrt{2\sqrt{n}}}$ is an integer. Find the sum of the digits of $n$.

(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

15. $*$ Let $x, y$ and $z$ be positive integers such that

$$x + \frac{1}{y + \frac{1}{z}} = \frac{26}{21}.$$

What is the value of $xyz$?

(A) 20 (B) 24 (C) 28 (D) 32 (E) 36

16. $*$ Twelve marbles are placed in three boxes such that each box contains a red marble, a blue marble, a black marble, and a white marble. If we pick at random one marble from each box, what is the probability that exactly two marbles are red?

(A) $\frac{9}{64}$ (B) $\frac{1}{8}$ (C) $\frac{3}{32}$ (D) $\frac{5}{64}$ (E) $\frac{1}{16}$

17. How many permutations $(x_1, x_2, x_3, x_4)$ of the set of integers $\{1, 2, 3, 4\}$ have the property that the sum $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ is not divisible by 3?

(A) 6 (B) 8 (C) 12 (D) 14 (E) 16

18. The numbers $a$ and $b$ are chosen from the set $\{1, 2, \ldots, 26\}$, such that the product $ab$ is equal to the sum of the remaining numbers. What is the value of $|a - b|$?

(A) 3 (B) 6 (C) 8 (D) 10 (E) 13

19. How many triangles $\Delta ABC$ with $\angle ABC = 90^\circ$ and $AB = 10$ exist such that all sides have integer lengths?

(A) infinitely many (B) 3 (C) 2

(D) 1 (E) none

20. Find the product of the solutions of the equation

$$\sqrt[3]{(x+1)^2} + \sqrt[3]{(x-1)^2} = \frac{5}{2}\sqrt[3]{x^2-1}.$$

(A) $-\dfrac{36}{121}$ (B) $-\dfrac{16}{81}$ (C) $-\dfrac{9}{25}$ (D) $-\dfrac{25}{36}$ (E) $-\dfrac{81}{49}$

21. $*$ Find the number of integers $n$ such that $\dfrac{n^2}{n+2018}$ is an integer.

(A) 12 (B) 14 (C) 16 (D) 18 (E) 20

22. What is the remainder when $2017^{2018}$ is divided by 13?

(A) 1 (B) 5 (C) 3 (D) 2 (E) 4

23. Three regular dice are rolled. The probability that the numbers thrown have the least common multiple equal to 60 is equal to $\dfrac{m}{n}$, for some relatively prime positive integers $m$ and $n$. What is the value of $\dfrac{n-2m}{m+3}$ ?

(A) 5 (B) 2 (C) 4 (D) 1 (E) 3

24. Vlad is playing on the mall escalators. One escalator goes up, one goes down, and one is out of service; otherwise, they're all identical. The up and down escalators go at the same speed. You can assume that Vlad always runs at the same speed. Vlad can run up the up escalator in 6 seconds. He can run up the down escalator in 30 seconds. How long does it take him to run up the out-of-service escalator?

(A) 10 s (B) 12 s (C) 14 s (D) 16 s (E) 18 s

25. How many triples $(x, y, z)$, of integer numbers, $x > y > z > 1$, satisfy the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

26. $*$ The function $f$ satisfies the relations

$$f(1) = 1 \text{ and } f(m+n) = f(m) + f(n) + mn,$$

for all positive integers $m$ and $n$. What is the value of $f(200)$?

(A) 20000 (B) 20010 (C) 20100 (D) 21000 (E) 21100

27. $*$ If $x, y$, and $z$ are real numbers such that $x, y, z > 1$, what is the smallest possible value of

$$\log_{xy} z + \log_{yz} x + \log_{xz} y?$$

(A) $\frac{7}{5}$ (B) $\frac{3}{2}$ (C) $\frac{8}{5}$ (D) $\frac{29}{20}$ (E) $\frac{4}{3}$

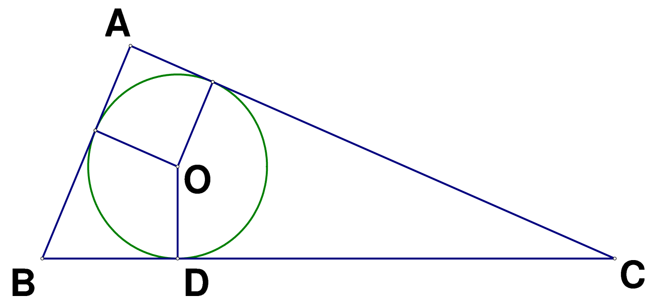
28. Find the sum of all real numbers $m$ with the property that the equation

$$x^2 - x + m = 0$$

has two solutions, $x_1$ and $x_2$, which satisfy the equation $x_1^5 + x_2^5 = 211$.

(A) 2 (B) 4 (C) 3 (D) 1 (E) 5

29. In the accompanying figure, $\Delta ABC$ is a right triangle with hypothenuse $BC = 2018$ cm and $AC = 1680$ cm. The circle inscribed in this triangle has radius $r = OD$. What is the value of $r$, in centimeters?




(A) 390 (B) 380 (C) 370

(D) 360 (E) 350

30. $*$ The function $f$ satisfies the conditions $f(4) = 6$ and $xf(x) = (x-3)f(x+1)$, for all integers $x$. What is the value of the product

$$f(2018) \cdot f(2015) \cdot f(2012) \cdots f(8) \cdot f(5)?$$

(A) 0 (B) 2017 (C) 2017! (D) 2018 (E) 2018!

31. Let $x$ and $y$ be positive integers such that $\dfrac{2018!}{7^x \cdot 13^y}$ is an integer. What is the largest possible value of $x + y$?

(A) 480 (B) 485 (C) 490 (D) 495 (E) 500

32. Find the sum of the digits of the unique solution of the equation

$$x^{\log_{100}(x-1)} + 100(x-1)^{\log_{100} x} = 101x^2.$$

(A) 2 (B) 6 (C) 10 (D) 14 (E) 18

33. $*$ If $x$ is a positive real number, what is the minimum possible value of the expression

$$E(x) = \frac{\left(x+\frac{1}{x}\right)^4 - \left(x^4+\frac{1}{x^4}\right)}{\left(x+\frac{1}{x}\right)^3 - \left(x^3+\frac{1}{x^3}\right)}?$$

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{5}{3}$ (D) $\frac{7}{3}$ (E) $\frac{8}{3}$

34. $*$ Let $x, y, z$ be real numbers such that

$$3^x + 3^y + 3^z + 3^{-x} + 3^{-y} + 3^{-z} = 10.$$

If $E(x, y, z) = 3^x + 3^y + 3^z$, what is the difference between the largest and the smallest possible values of $E(x, y, z)$?

(A) 8 (B) 7 (C) 9 (D) 6 (E) 10

35. $*$ Let $x$ be a real number such that

$$\sec x - \tan x = \frac{1}{2}.$$

Find the value of $\cos x$.

(A) $\frac{3}{4}$ (B) $\frac{2}{3}$ (C) $\frac{4}{5}$ (D) $\frac{6}{7}$ (E) $\frac{1}{2}$

36. $*$ A line is parallel to the line $y = \frac{5}{4}x + \frac{95}{4}$, intersects the $x$-axis and $y$-axis at points $A$ and $B$, respectively, and passes through the point $(-1, -25)$. How many points with integer coordinates are there on the line segment $AB$ (including points $A$ and $B$)?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

37. $*$ A sphere is inscribed in a cube, and a smaller cube is inscribed within the sphere. What is the ratio of the volume of the large cube to the volume of the small cube?

(A) $4\sqrt{2}$ (B) $6\sqrt{3}$ (C) $2\sqrt{2}$ (D) $2\sqrt{3}$ (E) $3\sqrt{3}$

38. Suppose that real numbers $x$ and $y$ satisfy the equation

$$4x^2 - 6xy + 4y^2 = 7.$$

Let $S = x^2 + y^2$ and $S_{min}$ and $S_{max}$ denote the minimum and maximum values of $S$, respectively. Find the value of $S_{min} + S_{max}$.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

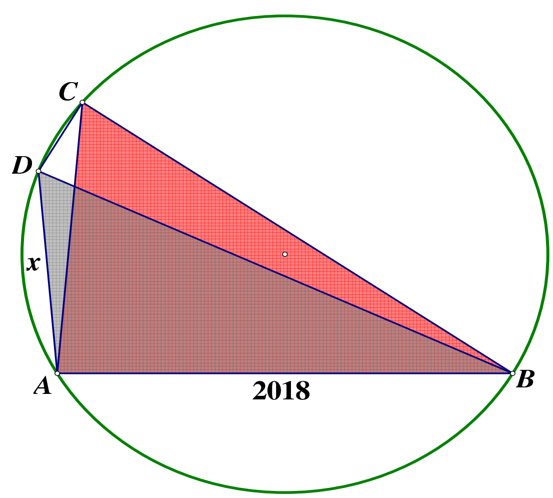
39. $*$ Two of the altitudes of a triangle are 10 cm and 11 cm. Which of the following **can not** be the length of the third altitude?

(A) 6 cm (B) 5 cm (C) 8 cm (D) 7 cm (E) 9 cm

40. In the accompanying figure, ABCD is a cyclic quadrilateral (inscribed in a circle) in which $AB = 2018$ cm, $BD = BC = 2322$ cm, and $AC = 1330$ cm. The length of $x = AD$ (in centimeters) is a three-digit number (in base 10). What is the units digit of $x$?

(A) 5 (B) 4 (C) 3

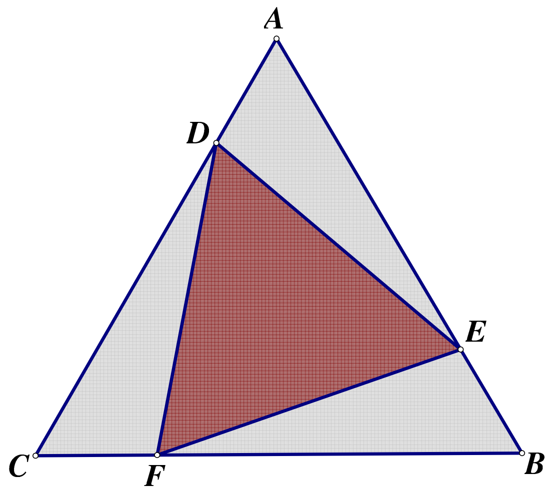
(D) 2 (E) 1



41. In the equilateral triangle $\Delta ABC$, the equilateral triangle $\Delta DEF$ is inscribed in such a way $\dfrac{CD}{DA} = \dfrac{AE}{EB} = \dfrac{BF}{CF} = 3$. The ratio between the areas of the triangles $\Delta ABC$ and $\Delta DEF$ is $\dfrac{m}{n}$, for some relatively prime positive integers $m$ and $n$. What is $5m - 11n$ ?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5




42. $*$ How many integer triples $(x, y, z)$ satisfy the equation

$$x^2 + y^2 + z^2 = 2^{2018}(x + y + z)?$$

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

43. For a positive integer $n$, written in base 10, we denote by $p(n)$ the product of its digits. What is the sum of the digits of $n$, if it satisfies the equation

$$10p(n) = n^2 + 6n - 2095?$$

(A) 15 (B) 12 (C) 6 (D) 18 (E) 9

44. Two positive numbers $a$ and $b$ are chosen in such a way that $\dfrac{b}{a} = e \approx 2.71828$ (the Euler number). Two points of coordinates $x$ and $y$ are chosen at random from the interval $[0, b]$. The probability that the geometric average of $x$ and $y$ is greater than $a$ is equal to $1 - \dfrac{m}{e^2}$. What is the value of $m$ ?

(A) 3 (B) 1 (C) 4 (D) 2 (E) 5

45. $*$ The sequence $\{x_n\}_{n\geq 1}$ is defined by the formula

$$x_n = \sin(\pi\sqrt{4n^2+2n+1}),$$

for all positive integers $n$. Find the value of $\lim_{n\to\infty} x_n$.

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{\sqrt{2}}{2}$ (D) 0 (E) $\frac{\sqrt{3}}{2}$

46. Consider the sequence $\{x_n\}_{n\geq 1}$ such that

$$x_1 = 1 \text{ and } x_{n+1} = x_n + \frac{1}{2x_n}, \text{ for } n \geq 1.$$

Find $\lim_{n\to\infty} \frac{x_n}{\sqrt{n}}$.

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

47. $*$ Let $a$ and $b$ be positive real numbers. If the equation

$$x + \ln(ab) = \ln(x+a) + \ln(x+b)$$

has only one real solution $x$, what is the value of $\frac{1}{a} + \frac{1}{b}$ ?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

48. Two points are chosen at random on a circle (uniform distribution on the circumference) of radius 1. The probability that the distance between them is more than 1 is equal to $\frac{m}{n}$, for some relatively prime positive integers $m$ and $n$. What is the value of $n-m$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

49. Two positive numbers $a$ and $b$ are chosen in such a way that $\frac{b}{a} = e \approx 2.71828$ (the Euler number). Two points of coordinates $x$ and $y$ are chosen at random from the

interval $[0, b]$. The probability that the geometric average of $x$ and $y$ is greater than $a$ is equal to $1 - \frac{m}{e^2}$. What is the value of $m$ ?

(A) 3 (B) 1 (C) 4 (D) 2 (E) 5

50. $*$ How many real numbers $x$, $0 \leq x \leq 2018$, are solutions of the equation

$$2^{\sin^4 x - \cos^2 x} - 2^{\cos^4 x - \sin^2 x} = \cos 2x?$$

(A) 1286 (B) 1285 (C) 1284 (D) 1283 (E) 1282