Solutions of the Thirty-seventh Annual Columbus State Invitational Mathematics Tournament

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1 A	6 C	11 B	16 B	21 D	26 E	31 A	36 A	41 D	46 B
2 D	7 E	12 C	17 A	22 A	27 A	32 A	37 D	42 E	47 E
3 C	8 D	13 E	18 D	23 C	28 C	33 D	38 C	43 A	48 B
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1. A student writes on the board seven consecutive integers. The sum of the smallest three is 36. Find the sum of the largest three.

   (A) 48 (B) 39 (C) 51 (D) 45 (E) 42

   Answer: A

   Denote by a the smallest number. The seven numbers are $a, a+1, a+2, a+3, a+4, a+5$, and $a+6$. Since the sum of the smallest three is 36 we have that $3a+3=36$. This implies $a=11$, so the sum of the largest three is 48.

2. Find one quarter of the number 16^{160}.

   (A) 4^{160} (B) 16^{40} (C) 8^{78} (D) 2^{638} (E) 4^{328}

   Answer: D

   We have $\frac{1}{4}\cdot 16^{160}=\frac{1}{4}\cdot 4^{320}=4^{319}=2^{638}$.

3. $*$ Find the solution of the equation $\dfrac{x-\sqrt{2}}{x-\sqrt{3}} = \dfrac{x-\sqrt{3}}{x-\sqrt{2}}$.

(A) $\sqrt{3}-\sqrt{2}$ (B) $\dfrac{2}{\sqrt{3}-\sqrt{2}}$ (C) $\dfrac{\sqrt{3}+\sqrt{2}}{2}$

(D) $\dfrac{2}{\sqrt{3}+\sqrt{2}}$ (E) $\sqrt{3}+\sqrt{2}$

Answer: C

Clearly $x \neq \sqrt{2}$ or $\sqrt{3}$. Cross multiply and solve the equation $x^2 - 2\sqrt{2}x + 2 = x^2 - 2\sqrt{3}x + 3$. It follows easily that $x = \dfrac{\sqrt{3}+\sqrt{2}}{2}$.

4. The lines $4y + 3x - 4 = 0$ and $6y - ax + 4 = 0$ are perpendicular. What is the value of a?

(A) -2 (B) 8 (C) 6 (D) -4 (E) 0

Answer: B

The slopes of these lines are $\dfrac{-3}{4}$ and $\dfrac{a}{6}$. For the product of the slopes to be -1 we must have $a = 8$.

5. $*$ The expression $\dfrac{4x^{-2} - 9y^{-2}}{a-b} \cdot \dfrac{a^{-2} - b^{-2}}{2x^{-1} + 3y^{-1}}$ is equivalent to which of the following?

(A) $\dfrac{(3x+2y)(a-b)}{xyab}$ (B) $\dfrac{(3x-2y)(a-b)}{a^2b^2}$ (C) $\dfrac{(3x+2y)(a+b)}{xy}$

(D) $\dfrac{(3x+2y)(a+b)}{x^2y^2ab}$ (E) $\dfrac{(3x-2y)(a+b)}{xya^2b^2}$

Answer: E

We have $\dfrac{4x^{-2} - 9y^{-2}}{a-b} \cdot \dfrac{a^{-2} - b^{-2}}{2x^{-1} + 3y^{-1}} = \dfrac{\frac{4}{x^2} - \frac{9}{y^2}}{a-b} \cdot \dfrac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{2}{x} + \frac{3}{y}} = \dfrac{4y^2 - 9x^2}{x^2y^2(a-b)} \cdot \dfrac{(b^2-a^2)xy}{a^2b^2(2y+3x)} = \dfrac{(2y-3x)(2y+3x)}{x^2y^2(a-b)} \cdot \dfrac{(b-a)(b+a)xy}{a^2b^2(2y+3x)} = \dfrac{(3x-2y)(a+b)}{xya^2b^2}$.

6. Find all solutions of the equation $\frac{1}{x-2}+\frac{1}{x+2}=\frac{8}{4-x^2}$.

(A) 4 (B) 0 (C) -4

(D) -4 and 4 (E) No solutions

Answer: C

The equation is equivalent to $\frac{2x}{x^2-4}=\frac{8}{4-x^2}$, which implies $x=-4$.

7. $*$ Let a and b be real numbers such that $a+b=5$ and $ab=1$. Which of the following is the quadratic equation with roots a and b?

(A) $5x^2+x+1=0$ (B) $x^2-x+5=0$ (C) $5x^2-x+1=0$

(D) $x^2+5x-1=0$ (E) $x^2-5x+1=0$

Answer: E

If a and b denote the roots of the quadratic equation $mx^2+nx+p=0$, then we have $a+b=-\frac{n}{m}$ and $ab=\frac{p}{m}$. In our case $a+b=5$ and $ab=1$, so the quadratic equation with roots a and b is $x^2-5x+1=0$.

8. Find the measure of each interior angle of a regular octagon.

(A) 115° (B) 100° (C) 90° (D) 135° (E) 130°

Answer: D

Using the center of the octagon as a common vertex we may divide the octagon into 8 congruent isosceles triangles. For each of these triangles the angle determined by the center of the octagon is 45°. This implies that the remaining angles in each triangle add up to 135°, so the measure of each interior angle of the octagon is 135°.

9. $*$ Let $0<a<b$ be two real numbers such that $a^2+b^2=10ab$. Find the value of $\frac{a+b}{a-b}$.

(A) $-\frac{\sqrt{6}}{3}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{2\sqrt{2}}{3}$ (E) $-\frac{\sqrt{6}}{2}$

Answer: E

We have $(a+b)^2 = 12ab$ and $(a-b)^2 = 8ab$, so $\left(\dfrac{a+b}{a-b}\right)^2 = \dfrac{3}{2}$. Since $0 < a < b$ we get $\left(\dfrac{a+b}{a-b}\right) = -\dfrac{\sqrt{6}}{2}$.

10. Find the sum of the solutions of the equation $(\log_3 x)^2 - 4\log_3 x + 3 = 0$.

(A) 20 (B) 40 (C) 30 (D) 60 (E) 50

Answer: C

The equation is equivalent to $(\log_3 x - 3)(\log_3 x - 1) = 0$. This implies $x = 27$ and $x = 3$, so the sum of the solutions is 30.

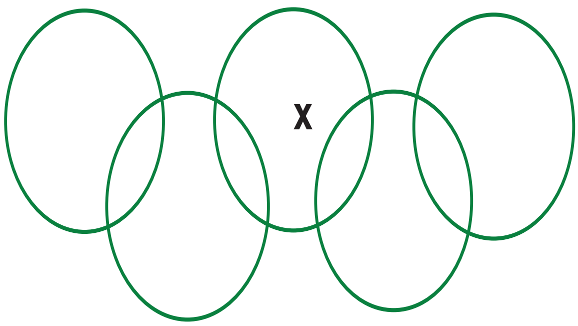
11. For each positive integer n we denote by $s(n)$ the sum of the digits of n. Find the smallest possible value of the ratio $\dfrac{n}{s(n)}$, for $10 \le n \le 30$.

(A) 1.7 (B) 1.9 (C) 2 (D) 1.8 (E) 1.6

Answer: B

For $10 \le n \le 30$, the possible values for $\dfrac{n}{s(n)}$ are in the set $\left\{10, \dfrac{11}{2}, 4, \dfrac{13}{4}, \dfrac{14}{5}, \dfrac{5}{2}, \dfrac{16}{7}, \dfrac{17}{8}, 2, \dfrac{19}{10}, 7, \dfrac{11}{2}, \dfrac{23}{5}, \dfrac{25}{7}, \dfrac{13}{4}, 3, \dfrac{14}{5}, \dfrac{29}{11}\right\}$, so the smallest value is 1.9.

12. $*$ The five intersecting circles in the figure determine nine regions. In each of these regions you write one of the numbers from 1 to 9 such that each number is written exactly once and the sum of the numbers inside each circle is 11. Which number must be written in the region marked with an **X**?



(A) 7 (B) 4 (C) 6

(D) 3 (E) 5

Answer: C

Note that numbers 8 and 9 can only be written inside the regions most to the left or right. Also, note that the numbers 1, 2, 3, and 4 can only be written in the small regions determined by exactly two intersecting circles. Using these two observations it follows that only 6 can be written in the region marked by **X**.

13. $*$ The sum of 2011 consecutive integers is 2011. Find the sum of the digits of the largest of these integers.

(A) 5 (B) 6 (C) 4 (D) 8 (E) 7

Denote by a the smallest of these integers. We have $a+(a+1)+(a+2)+\cdots+(a+2010) = 2011$. This implies that $2011a+\frac{2010\cdot 2011}{2} = 2011$, so $a = -1004$. The largest integer of the sum is $a+2010 = 1006$, so the sum of its digits is 7.

14. Let x be a real number such that $x+\frac{1}{x} = 4$. Find the value of $x^3+\frac{1}{x^3}$.

(A) 40 (B) 44 (C) 48 (D) 52 (E) 56

Answer: D

Using the binomial formula we get $\left(x+\frac{1}{x}\right)^3 = x^3+3x^2\cdot\frac{1}{x}+3x\cdot\frac{1}{x^2}+\frac{1}{x^3}$. This implies that $x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right) = 64-12 = 52$.

15. $*$ Find the sum $\frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}}+\frac{\sqrt{3}-\sqrt{2}}{\sqrt{6}}+\cdots+\frac{\sqrt{2011}-\sqrt{2010}}{\sqrt{2010\cdot 2011}}$.

(A) $\frac{\sqrt{2010}}{\sqrt{2011}}$ (B) $\frac{1}{\sqrt{2011}}$ (C) $\frac{\sqrt{2011}-1}{\sqrt{2011}}$

(D) $\sqrt{2011}$ (E) $\frac{\sqrt{2011}-\sqrt{2010}}{\sqrt{2011}}$

Answer: C

If we distribute the denominator we get the equivalent telescopic sum $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{2010}}-\frac{1}{\sqrt{2011}} = 1-\frac{1}{\sqrt{2011}} = \frac{\sqrt{2011}-1}{\sqrt{2011}}$.

16. Find the number of elements of the set $A = \{2,3,6,7,10,11,14,15,\ldots,2006,2007,2010,2011\}$.

(A) 1004 (B) 1006 (C) 1008 (D) 1010 (E) 1012

Answer: B

The pattern describing the set suggest that the integers "missing" are of the form $4k$ or $4k+1$. There are 502 integers of the form $4k$ between 1 and 2011. Similarly, there and 503 integers of the form $4k+1$ between 1 and 2011. This implies that the set has 2011-1005=1006 elements.

17. $*$ The positive integers x and y satisfy the relation $3x = 7y$. Which of the following is a possible value for $x + y$?

(A) 2010 (B) 1005 (C) 201 (D) 335 (E) 402

Answer: A

We have $x = \frac{7y}{3}$, so $x + y = \frac{10y}{3}$. Because $x + y$ is an integer and $x + y = \frac{10y}{3}$, we conclude that $x + y$ has to be a multiple of 10. Therefore, the only possible value for $x + y$ is 2010.

18. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and m and n real numbers such that

$$mf(x-1) + nf(-x) = 2|x| + 1$$

for all real numbers x. If $f(-2) = 5$ and $f(1) = 1$, then find the numbers m and n.

(A) $m = \frac{1}{6}; n = \frac{5}{6}$ (B) $m = \frac{3}{4}; n = \frac{7}{4}$ (C) $m = \frac{3}{5}; n = \frac{8}{5}$

(D) $m = \frac{5}{12}; n = \frac{11}{12}$ (E) $m = \frac{2}{5}; n = 1$

Answer: D

Substitute $x = -1$ and then $x = 2$ in the above relation. We get the system of equations

$$\begin{cases} 5m + n = 3 \\ m + 5n = 5 \end{cases}$$

It follows easily that $m = \frac{5}{12}$ and $n = \frac{11}{12}$.

19. $*$ Find the quadratic function $f(x) = ax^2 + bx + c$ knowing that for every positive integer n we have $f(1) + f(2) + \cdots + f(n) = n^3$.

(A) $f(x) = x^2 - 4x + 4$ (B) $f(x) = 4x^2 + 2x - 3$ (C) $f(x) = 5x^2 - 3x + 2$

(D) $f(x) = 3x^2 - 3x + 1$ (E) $f(x) = x^2 + 3x - 3$

Answer: D

Using that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ we get that $f(1) + f(2) + \cdots + f(n) = a \cdot \frac{n(n+1)(2n+1)}{6} + b \cdot \frac{n(n+1)}{2} + c \cdot n$. This implies that we have the identity $a \cdot \frac{n(n+1)(2n+1)}{6} + b \cdot \frac{n(n+1)}{2} + c \cdot n = n^3$, for all positive integers n. Divide by n and combine similar terms to get the equivalent identity $2an^2 + n(3a + 3b) + a + 3b + 6c = 6n^2$, for all positive integers n. It follows that $2a = 6$, $3a + 3b = 0$, and $a + 3b + 6c = 0$, so $a = 3$, $b = -3$, $c = 1$, and $f(x) = 3x^2 - 3x + 1$.

20. ∗ How many distinct permutations of the letters of the word "**PEPPER**" are there?

(A) 720 (B) 40 (C) 360 (D) 120 (E) 60

Answer: E

The number of distinct ways we can permute the letters P of the word "**PEPPER**" is $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$. For each choice of the three P's there are 3 possibilities to permute the letter R. For each choice of the P's and the R there is exactly 1 possibility to arrange the remaining two letters E. Therefore, the number of distinct permutations is $20 \cdot 3 \cdot 1 = 60$.

21. ∗ Find the number of positive integers n, $1 \leq n \leq 100$, such that n^n is a perfect square.

(A) 53 (B) 52 (C) 50 (D) 55 (E) 51

Answer: D

A positive integer $n \geq 2$ is a perfect square if and only if the exponent of each prime, in its prime factorization, is even. If n is even then clearly all primes in the prime factorization of n^n have an even exponent. Since $1 \leq n \leq 100$, we have 50 even numbers such that n^n is a perfect square. If n is odd then the only way that n^n is a perfect square is if n is a square. (If n was not a perfect square then there would be prime number with an odd exponent in its prime factorization, so n^n would have a prime number with an odd exponent as well). Since $1 \leq n \leq 100$, we get that the only odd numbers such that n^n is a perfect square are $n = 1, 9, 25, 49, 81$. Thus, there are 55 numbers with the required property.

22. For n positive integer, let $S_n = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n}}$.
Find the smallest value of n such that $S_n \geq 100$.

(A) 10,200 (B) 10,000 (C) 9,400 (D) 9,600 (E) 9,800

Answer: A

Note that $S_n = \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots + \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - 1$. This implies that we want to find the smallest value of n such that $\sqrt{n+1} - 1 \geq 100$. The inequality is equivalent to $n \geq 10,200$, so the smallest value of n is $10,200$.

23. $*$ Find the number of pairs of integers (x, y) which satisfy the equation

$$\frac{1}{x+2010} + \frac{2011}{y+2011} = 1.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: C

Clearly $x \neq -2010$ and $y \neq -2011$. Rewrite the equation as $\frac{1}{x+2010} = \frac{y}{y+2011}$ and cross multiply. We get $y + 2011 = y(x + 2010)$. Since x and y are integers, the last relation implies that y is a divisor of 2011. To see this note that y divides $(y + 2011)$ and since, obviously, y divides y we have that y divides $(y + 2011 - y)$, so y divides 2011. Because 2011 is prime we obtain $y = \pm 1$ or $y = \pm 2011$. Since $y \neq -2011$ we get the solutions $(x, y) = (2, 1)$, $(x, y) = (-2008, 2011)$, and $(x, y) = (-4020, -1)$, so there are 3 pairs satisfying the given equation.

24. Which is the coefficient of x^{35} in the binomial expansion of $(1 + x^5)^{10}$?

(A) 120 (B) 210 (C) 250 (D) 320 (E) 370

Answer: A

Using the binomial formula we get $(1 + x^5)^{10} = \sum_{i=0}^{10} \binom{10}{i} x^{5i}$. The coefficient of x^{35} is obtained for $i = 7$. It is $\binom{10}{7} = 120$.

25. How many pairs (x, y) of positive integers satisfy $x^2 + y^2 = 2011$?

(A) 2011 (B) 2 (C) 0 (D) 1005 (E) 10

Answer: C

An elementary result in Number Theory states that a prime number of the form $4k+3$ divides the sum of two squares if and only if it divides each of them. Assume that

(x, y) is a solution of the equation $x^2 + y^2 = 2011$, with x, y integers. Since 2011 is of the form $4k + 3$ we have that 2011 divides both x and y. Therefore, $x = 2011a$ and $y = 2011b$, for some integers a and b. The equation becomes $2011^2(a^2 + b^2) = 2011$, so we get $2011(a^2 + b^2) = 1$, which is impossible for a, b integers. Thus, there is no solution in integers for the equation $x^2 + y^2 = 2011$.

26. The number 2011 is written as the sum of $(n + 1)$ consecutive integers

$$2011 = a + (a + 1) + (a + 2) + \cdots + (a + n).$$

If $1 < n < 2011$, then find the value of a.

(A) -62 (B) -250 (C) -125 (D) -502 (E) -1004

Answer: E

We have $2011 = (n+1)a + \dfrac{n(n+1)}{2}$, so we get $4022 = (n+1)(2a+n)$. Because $(n+1)$ divides 4022 and $1 < n < 2011$, we get that $n + 1 = 2011$, so $n = 2010$. This implies $2a + 2010 = 4022$, so $a = -1004$.

27. $*$ Find the number of integers n such that $\dfrac{n^2 + 2}{n + 1}$ is an integer.

(A) 4 (B) 0 (C) 6 (D) 2 (E) 8

Answer: A

Clearly $n \neq -1$. Note that $\dfrac{n^2 + 2}{n + 1} = n - 1 + \dfrac{3}{n + 1}$. This implies that $\dfrac{n^2 + 2}{n + 1}$ is an integer if and only if $\dfrac{3}{n + 1}$ is an integer. It follows that $n + 1 = \pm 1$ or $n + 1 = \pm 3$, so $n = 0, -2, -4, 2$. None of these values is equal to -1, so there are 4 solutions.

28. $*$ Find the integers a and b such that $\dfrac{a}{\sqrt{3 + 2\sqrt{2}}} + \dfrac{b}{\sqrt{3 - 2\sqrt{2}}} = \sqrt{11 + 6\sqrt{2}}$.

(A) $a = 2$, $b = -1$ (B) $a = -2$, $b = 1$ (C) $a = -1$, $b = 2$

(D) $a = 1$, $b = -2$ (E) $a = 1$, $b = 2$

Answer: C

The equation $\dfrac{a}{\sqrt{3+2\sqrt{2}}}+\dfrac{b}{\sqrt{3-2\sqrt{2}}}=\sqrt{11+6\sqrt{2}}$ is equivalent to $\dfrac{a}{\sqrt{\left(\sqrt{2}+1\right)^2}}+\dfrac{b}{\sqrt{\left(\sqrt{2}-1\right)^2}}=\sqrt{\left(3+\sqrt{2}\right)^2}$. This observation implies that we have to solve $\dfrac{a}{\sqrt{2}+1}+\dfrac{b}{\sqrt{2}-1}=3+\sqrt{2}$. The last equation may be written $\sqrt{2}(a+b-1)=a-b+3$. Since a and b are integers we need to have $a+b-1=0$. (Otherwise the equation has no solutions. This is because if $a+b-1\neq 0$ we get $\sqrt{2}=\dfrac{a-b+3}{a+b-1}$, an equation with no integer solutions a and b because $\sqrt{2}$ is irrational.)

Since $a+b-1=0$, we get that $a-b+3=0$, so $a=-1$ and $b=2$.

29. Find the values of the real number a such that the equations $x^2+x+a=0$ and $ax^2+x+1=0$ have exactly one common root.

(A) -1 and 0 (B) -1 and 2 (C) 0 and 2

(D) -2 and 2 (E) -2 and 0

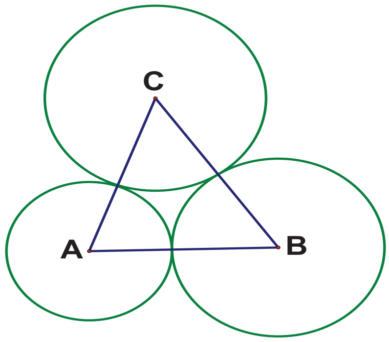
Answer: E

Any common root for $x^2+x+a=0$ and $ax^2+x+1=0$ is a root for $x^2(1-a)+a-1=0$, the equation obtained by subtracting the first two. The last equation can be written in the form $(1-a)(x^2-1)=0$. Note that if the two equations are to have exactly one common root then $a\neq 1$. This is because $a=1$ would make the two equations coincide, so they would have in common two distinct (complex) roots. This implies that any common root for the two equations must satisfy $x^2-1=0$, so $x=\pm 1$. If $x=1$ is a common root, then $1^2+1+a=0$, so $a=-2$. Thus, the two equations are $x^2+x-2=0$ and $-2x^2+x+1=0$. The first one has roots 1 and -2 and the second 1 and $-\frac{1}{2}$, so $a=-2$ satisfies the required property. If $x=-1$ is a common root, then $a=0$, so the two equations are $x^2+x=0$ and $x+1=0$. Clearly $a=0$ satisfies the required property. In conclusion, $a=-2$ and $a=0$ are the values for which the equations have exactly one common root.

30. ∗ Three circles with radii 6, 7, and 8 and centers A, B, and C are tangent to each other as in the figure. Find the area of the triangle ΔABC.

(A) 48 (B) 60 (C) 72

(D) 84 (E) 96



Answer: D

Note that $AB = 13$, $AC = 14$, and $BC = 15$, so the semi-perimeter of the triangle ΔABC is $p = 21$. We use Heron's formula for the area of a triangle

$$Area_{\Delta ABC} = \sqrt{p(p-BC)(p-AC)(p-AB)} = \sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84.$$

31. ∗ Solve the inequality $\sin x + \sin 2x \geq 0$, for x in $[0, 2\pi)$.

(A) $\left[0, \frac{2\pi}{3}\right] \cup \left[\pi, \frac{4\pi}{3}\right]$ (B) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right)$ (C) $\left[0, \frac{7\pi}{12}\right] \cup \left[\pi, \frac{17\pi}{12}\right]$

(D) $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$ (E) $[0, \pi] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

Answer: A

Using the double angle formula $\sin 2x = 2\sin x \cos x$ we get the equivalent equation

$$\sin x(1 + 2\cos x) \geq 0$$

Now, $\sin x \geq 0$ for x in $[0, \pi]$ and $\sin x < 0$ for x in $(\pi, 2\pi)$.

Also, $1 + 2\cos x \geq 0$ for x in $\left[0, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right)$ and $1 + 2\cos x < 0$ for x in $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$.

Combining these inequalities we get that x is in $\left[0, \frac{2\pi}{3}\right] \cup \left[\pi, \frac{4\pi}{3}\right]$.

32. Let a, b, and c be positive real numbers such that $\dfrac{2a}{b+4c} = \dfrac{b}{2a+4c} = \dfrac{4c}{2a+b}$. Find the value of $\left(\dfrac{1}{3a} + \dfrac{1}{b} + \dfrac{1}{4c}\right)(3a+b+4c)$.

(A) $\dfrac{28}{3}$ (B) $\dfrac{35}{3}$ (C) $\dfrac{49}{3}$ (D) $\dfrac{56}{3}$ (E) $\dfrac{63}{3}$

Answer: A

Using the properties of proportions and $4a + 2b + 8c \neq 0$ we get

$$\frac{2a}{b+4c} = \frac{b}{2a+4c} = \frac{4c}{2a+b} = \frac{2a+b+4c}{4a+2b+8c} = \frac{1}{2}$$

This implies $\begin{cases} 4a = b + 4c \\ 2b = 2a + 4c \\ 8c = 2a + b \end{cases}$

Combining the first two equations we get $4a - 2b = b - 2a$, so $b = 2a$, which implies $a = 2c$. This leads to

$$\left(\frac{1}{3a} + \frac{1}{b} + \frac{1}{4c}\right)(3a+b+4c) = \left(\frac{1}{3a} + \frac{1}{2a} + \frac{1}{2a}\right)(3a+2a+2a) = \frac{4}{3} \cdot 7 = \frac{28}{3}.$$

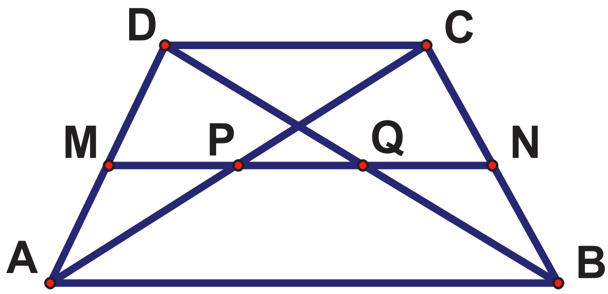
33. Find the number of all positive integers n such that $N = (n^2+6)^2 - (4n+2)^2$ is a prime number.

(A) 0 (B) 6 (C) 10 (D) 2 (E) 8

Answer: D

Note that $N = (n^2-4n+4)(n^2+4n+8) = (n-2)^2(n^2+4n+8)$. Since $n^2+4n+8 > 1$ and $(n-2)^2$ is a square for all positive integers n, the only way that N can be a prime is if $(n-2)^2 = 1$ and n^2+4n+8 is prime. This leads to $n = 3$ and $n = 1$. We get $N = 29$ and $N = 13$.

34. $*$ Let $ABCD$ be an isosceles trapezoid such that $\overline{AB} \parallel \overline{CD}$, $AD = BC = DC = 18$, and $m\angle A = 60°$. Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$. The segment $\overline{MN}$ intersects the diagonal $\overline{AC}$ at P and the diagonal $\overline{BD}$ at Q. Find the length of the segment $\overline{PQ}$.



(A) 7 (B) 9 (C) 8

(D) 10 (E) 6

Answer: B

Denote by D' and C' the projections of D and C on $\overline{AB}$. Since $m\angle A = 60°$, in the right triangle $\Delta DAD'$ we have $\cos 60° = \frac{AD'}{AD}$, so $AD' = 9$. By reasons of symmetry we get that $BC' = 9$. This implies that $AB = 36$. Since $MN = \frac{AB + CD}{2}$, we get that $MN = 27$. In addition, in the triangle ΔACD we have $MP = \frac{DC}{2} = 9$ because M and P are the midpoints of sides $\overline{AD}$ and $\overline{AC}$. Similarly, by symmetry we have $NQ = 9$. Finally, since $MN = MP + PQ + QN$ we get $27 = 9 + PQ + 9$, so $PQ = 9$.

35. Let ΔABC be a triangle with sides $AB = 10$, $AC = 24$, and $BC = 26$. Find the length of the median corresponding to $\overline{BC}$.

(A) 12 (B) 13 (C) 5 (D) 17 (E) 7

Answer: B

Note that $BC^2 = AB^2 + AC^2$, so ΔABC is a right triangle. In any right triangle the median corresponding to the hypotenuse has half the length of the hypothenuse, so the answer is 13.

36. $*$ Three balls are randomly drawn, without replacement, from a bowl containing 6 yellow and 5 blue balls. Find the probability that one of the balls is yellow and the other two are blue.

(A) $\frac{4}{11}$ (B) $\frac{3}{11}$ (C) $\frac{7}{11}$ (D) $\frac{5}{11}$ (E) $\frac{6}{11}$

Answer: A

Denote the probability for an "event" to happen $\mathbf{P}$(event). With this notation, the desired probability is $\mathbf{P}$(drawing 1 yellow and 2 blue) and is equal to $\mathbf{P}$(drawing first yellow, second and third blue)+$\mathbf{P}$(drawing first blue, second yellow, third blue)+$\mathbf{P}$(drawing fist and second blue, third yellow)=$\frac{6}{11}\cdot\frac{5}{10}\cdot\frac{4}{9}+\frac{5}{11}\cdot\frac{6}{10}\cdot\frac{4}{9}+\frac{5}{11}\cdot\frac{4}{10}\cdot\frac{6}{9}=\frac{4}{11}$.

37. Three constructors built 100 houses in a new subdivision. Constructor A built 20 houses, B built 30, and C built 50. The probabilities of developing a leaky basement are 10%, 8%, and 3% for constructors A, B, and C, respectively. Suppose that one basement is leaking. Find the probability that the house was built by constructor A.

(A) 0.059 (B) 0.041 (C) 0.41 (D) 0.34 (E) 0.25

Answer: D

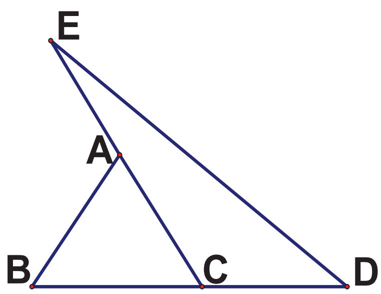
Denote by $\mathbf{P}(A|L)$ the conditional probability that A built the house with the leaking basement. Then we have

$$\mathbf{P}(A|L)=\frac{\mathbf{P}(A\text{and}L)}{\mathbf{P}(A\text{and}L)+\mathbf{P}(B\text{and}L)+\mathbf{P}(C\text{and}L)},$$

where $\mathbf{P}(A\text{and}L)$, $\mathbf{P}(B\text{and}L)$, and $\mathbf{P}(C\text{and}L)$ are the probabilities that A, B, and C respectively build a house with a leaking basement.

We have $\mathbf{P}(A\text{and}L)=\frac{20}{100}\cdot\frac{10}{100}=\frac{2}{100}$, $\mathbf{P}(B\text{and}L)=\frac{30}{100}\cdot\frac{8}{100}=\frac{24}{1000}$, and $\mathbf{P}(C\text{and}L)=\frac{50}{100}\cdot\frac{3}{100}=\frac{15}{1000}$, so $\mathbf{P}(A|L)=\frac{2}{100}\cdot\frac{1000}{59}=\frac{20}{59}\approx 0.34$.

38. ∗ An equilateral triangle ΔABC has sides equal to 10. Let D be a point on the line passing through B and C such that BC=CD, and let E be a point on the line passing through C and A such that CA=AE. Find the length of the segment ED.



(A) $11\sqrt{6}$ (B) $12\sqrt{5}$ (C) $10\sqrt{7}$

(D) $8\sqrt{10}$ (E) $7\sqrt{13}$

Answer: C

By the Law of Cosines in the triangle ΔCED we have that

$$ED^2=EC^2+CD^2-2EC\cdot CD\cdot\cos 120^\circ=700$$

This implied $ED = 10\sqrt{7}$.

39. If two fair dice are tossed, then find the least number of throws for which the probability of getting at least one double five exceeds 0.5?

(A) 27 (B) 14 (C) 25 (D) 23 (E) 19

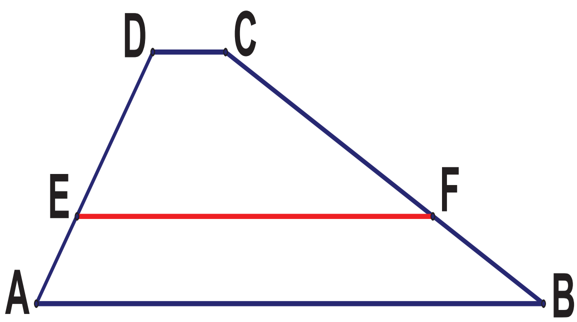
Answer: C

The probability of getting a double 5 in k throws is $1 - \left(\frac{35}{36}\right)^k$. Thus, we want the smallest k such that $1 - \left(\frac{35}{36}\right)^k \geq \frac{1}{2}$. This implies $k \ln \frac{35}{36} < \ln \frac{1}{2}$, so $k \geq 25$.

40. ∗ The trapezoid $ABCD$ has bases $AB = 2814$ and $CD = 402$. The segment $\overline{EF}$ is parallel to the bases and divides $ABCD$ in two regions with equal area. Find the length of $\overline{EF}$.

(A) 1608 (B) 2010 (C) 2412

(D) 804 (E) 1206



Answer: B

Denote by h the length of the height of the trapezoid $ABCD$, by a the length of $\overline{EF}$, by x the height of the trapezoid $DCEF$, and by $\mathbb{A}$ the area of the trapezoid $ABCD$.

Then we have $\begin{cases} \mathbb{A} = 1608h \\ \mathbb{A} = (402 + a)x \\ \mathbb{A} = (a + 2814)(h - x) \end{cases}$

These relations imply $\frac{h - x}{x} = \frac{a + 402}{a + 2814}$ and $\frac{h}{x} = \frac{402 + a}{1608}$. Thus we get

$$\frac{a + 402}{a + 2814} + 1 = \frac{402 + a}{1608}.$$

Common denominator and cross multiplication reduce the equation to

$$(2a + 3216) \cdot 1608 = (402 + a)(a + 2814).$$

This equation is equivalent to $a^2 = 3216 \cdot 1608 - 402 \cdot 2814$. That is $a^2 = 2010^2$, so $a = 2010$.

41. $*$ Let a, b, c, and d be positive real numbers such that $ac = bd = 10$. Find the smallest possible value of the expression $E = (a+2)(b+2)(c+5)(d+5)$.

(A) 1000 (B) 1200 (C) 1400 (D) 1600 (E) 1800

Answer: D

We have $E = (ac+5a+2c+10)(bd+5b+2d+10) = \left(20+5a+\frac{20}{a}\right)\left(20+5b+\frac{20}{b}\right)$.
Consider the function $f : (0,\infty) \to \mathbb{R}, f(x) = 20+5x+\frac{20}{x}$. The derivative of the function is $f'(x) = 5 - \frac{20}{x^2}$, so the function has a critical point at $x = 2$. Since $f'(x) < 0$ on $(0,2)$ and $f'(x) > 0$ on $(2,\infty)$, we get that the function f is decreasing on $(0,2)$ and increasing on $(2,\infty)$. It follows easily that $x = 2$ is an absolute minimum on $(0,\infty)$, so the smallest possible value for $f(x) = 20+5x+\frac{20}{x}$ is $f(2) = 40$. This implies that the smallest possible value for $E = \left(20+5a+\frac{20}{a}\right)\left(20+5b+\frac{20}{b}\right)$ is 1600. (when $a = b = 2$ and $c = d = 5$)

42. Assume $x > 0$. Which is the correct formula for $\arctan(x) + \arctan\left(\frac{1}{2x}\right)$?

(A) $\arctan\left(x+\frac{1}{x}\right)$ (B) $\arctan\left(x+\frac{1}{2x}\right)$ (C) $\arctan\left(2x+\frac{1}{2x}\right)$

(D) $\arctan\left(x+\frac{2}{x}\right)$ (E) $\arctan\left(2x+\frac{1}{x}\right)$

Answer: E

For $x > 0$, we have $\arctan x + \arctan\frac{1}{2x} = \arctan\left(\frac{x+\frac{1}{2x}}{1-x\cdot\frac{1}{2x}}\right) = \arctan\left(2x+\frac{1}{x}\right)$.

43. Find the positive real number a such that $x^a \le a^x$, for all positive real numbers x.

(A) e (B) π (C) $\sqrt{e}$ (D) $\pi - \frac{1}{2}$ (E) $e + \frac{1}{2}$

Answer: A

Apply "ln" to the inequality $x^a \le a^x$ and get the equivalent inequality $\frac{\ln x}{x} \le \frac{\ln a}{a}$, for all $x > 0$. ("ln" is increasing.) Consider the function $f : (0,\infty) \to \mathbb{R}$, defined by the formula $f(x) = \dfrac{\ln x}{x}$. Finding the number a such that $x^a \leq a^x$ for all $x > 0$ is equivalent to finding the maximum of f on $(0, \infty)$. The function f is differentiable and $f'(x) = \dfrac{1 - \ln x}{x^2}$, so $x = e$ is a critical point. We have that $f'(x) > 0$ on $(0, e)$ and $f'(x) < 0$ on (e, ∞). This implies that f is increasing on $(0, e)$ and decreasing on (e, ∞). It follows easily that $x = e$ is an absolute maximum for f on $(0, \infty)$, so $a = e$.

44. $*$ Let $S = 1^{2011} + 2^{2011} + 3^{2011} + \cdots + 2010^{2011}$. Find the remainder of the division of S by 2011.

(A) 2010 (B) 0 (C) 2009 (D) 1 (E) 1005

Answer: B

We use Fermat's little theorem. This results states that if a in any integer and p any prime number then p divides $a^p - a$. It is equivalent to say that a^p and a have the same reminder modulo p.

Because 2011 is prime, we get that the reminder of the division of $S = 1^{2011} + 2^{2011} + \cdots + 2010^{2011}$ by 2011 is the same as the reminder of $1 + 2 + \cdots + 2010$ by 2011. Since $1 + 2 + \cdots + 2010 = \dfrac{2010 \cdot 2011}{2} = 1005 \cdot 2011$ we get that the reminder of the division by 2011 is 0.

45. $*$ Let be a function defined for all points $P(x, y)$ in the plane by the formula $f(P) = 23x - 27y$. If $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_{503}(x_{503}, y_{503})$ are the vertices of a regular polygon centered at the point $(26, 22)$, then find the value of $f(P_1) + f(P_2) + ... + f(P_{503})$.

(A) 8048 (B) 1006 (C) 6036 (D) 4024 (E) 2012

Answer: E

Note that the centroid of the polygon $(26, 22)$ is obtained by averaging the x and the y coordinates of its vertices. This means that we have $\dfrac{\sum_{i=1}^{503} x_i}{503} = 26$ and $\dfrac{\sum_{i=1}^{503} y_i}{503} = 22$. Now,

$$f(P_1) + f(P_2) + ... + f(P_{503}) = 23\sum_{i=1}^{503} x_i - 27\sum_{i=1}^{503} y_i = 23 \cdot 26 \cdot 503 - 27 \cdot 22 \cdot 503 = 2012.$$

46. Let x, y, and z be three positive real numbers such that $x + y + z = \sqrt{10 + \sqrt{19}}$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \sqrt{10 - \sqrt{19}}$. If $a = x^2 + y^2 + z^2$ and $b = a + \frac{9}{a}$, then find the value of b.

(A) 7 (B) $\frac{20}{3}$ (C) $\frac{22}{3}$ (D) $\frac{19}{3}$ (E) $\frac{23}{3}$

Answer: B

Since $x + y + z = \sqrt{10 + \sqrt{19}}$ we get $\frac{x + y + z}{3} = \frac{\sqrt{10 + \sqrt{19}}}{3}$. Also, the relation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \sqrt{10 - \sqrt{19}}$ implies that $\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} = \frac{\sqrt{10 + \sqrt{19}}}{3}$.

Note that the arithmetic mean of the positive numbers x, y, z is equal to the harmonic mean of x, y, z. This implies that $x = y = x = \frac{\sqrt{10 + \sqrt{19}}}{3}$. It follows that $a = 3x^2 = \frac{10 + \sqrt{19}}{3}$ and that $\frac{9}{a} = \frac{10 - \sqrt{19}}{3}$. Finally, $b = a + \frac{9}{a} = \frac{20}{3}$.

47. * Let α be the acute angle formed by the tangent lines to the graphs of $y = \sin x$ and $y = \cos x$ at $x = \frac{\pi}{4}$. Find $\cos \alpha$.

(A) $\frac{2}{5}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{5}$ (E) $\frac{1}{3}$

Answer: E

The graphs of $y = \sin x$ and $y = \cos x$ intersect at the point $C\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$. Because the derivative of $\sin x$ is $\cos x$, the slope of the tangent line to the graph of $y = \sin x$, at $\frac{\pi}{4}$, is $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Similarly, because the derivative of $\cos x$ is $-\sin x$, the slope of the tangent line to the graph of $y = \cos x$, at $x = \frac{\pi}{4}$, is $-\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2}$. Therefore, the equations of the tangent lines to the graphs of $y = \sin x$ and $y = \cos x$ at $x = \frac{\pi}{4}$ are

$$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

and

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

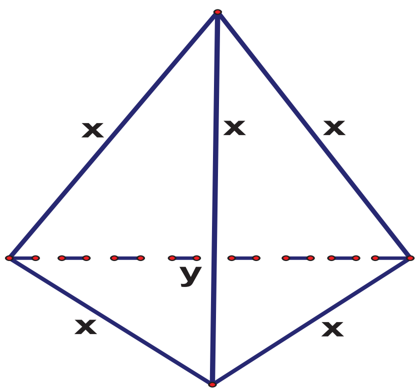
These lines intersect the y-axis at $A\left(0, \frac{\sqrt{2}(4-\pi)}{8}\right)$ and $B\left(0, \frac{\sqrt{2}(4+\pi)}{8}\right)$, respectively. We obtain the vectors $\overrightarrow{CA} = \left(\frac{\pi}{4}, \frac{\sqrt{2}\pi}{8}\right)$ along the tangent line to $y = \sin x$ and $\overrightarrow{CB} = \left(\frac{\pi}{4}, -\frac{\sqrt{2}\pi}{8}\right)$ along the tangent line to $y = \cos x$. Now, using the dot product and the length of the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$, we get

$$\cos\alpha = \frac{<\overrightarrow{CA}, \overrightarrow{CB}>}{\|\overrightarrow{CA}\| \cdot \|\overrightarrow{CB}\|} = \frac{\frac{\pi^2}{16} - \frac{\pi^2}{32}}{\frac{3\pi^2}{32}} = \frac{1}{3}.$$

48. ∗ One picks at random two numbers in the interval $[0, 1]$, first x and then y. What is the probability that a tetrahedron with side lengths as in the figure can be constructed?

(A) $1 - \frac{\sqrt{2}}{2}$ (B) $1 - \frac{\sqrt{3}}{6}$ (C) $1 - \frac{\sqrt{6}}{3}$

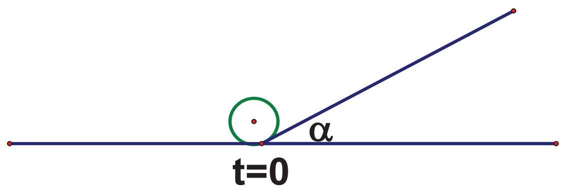
(D) $1 - \frac{\sqrt{6}}{4}$ (E) $1 - \frac{\sqrt{2}}{8}$



Answer: B

For a tetrahedron to be constructed we need to have $0 < y < \sqrt{3}x$. Clearly we need to have $0 < y$. To justify the other inequality, note that y should be smaller than the diagonal of the rhombus formed by "gluing", along a side, two equilateral triangles with all sides equal to x. For a given x, is it as if one stretches y until the 3 dimensional tetrahedron becomes a 2 dimensional rhombus. The long diagonal of this rhombus, with sides and a diagonal of length x, is $\sqrt{3}x$. The area of the region bounded by $0 < y < \sqrt{3}x$, $0 < x, y < 1$ is $1 - \frac{1}{2\sqrt{3}}$, so it follows that the probability required is $1 - \frac{\sqrt{3}}{6}$.

49. An object is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and that $t = 0$ is when the object enters the slope. If g denotes Earth's gravity, then find the time required for the object to return to its initial position at $t = 0$.



(A) $\dfrac{2v_0}{g\sin\alpha}$ (B) $\dfrac{v_0}{2g\sin\alpha}$ (C) $\dfrac{v_0}{g\sin\alpha}$

(D) $\dfrac{v_0}{g\cos\alpha}$ (E) $\dfrac{2v_0}{g\cos\alpha}$

Answer: A

The components of the force acting on the object are $F_x = m \cdot g\sin\alpha$ and $F_y = m \cdot g\cos\alpha$. In the absence of friction the acceleration is $\dfrac{F_x}{m} = -g\sin\alpha$, so the velocity is $v(t) = v_0 - (g\sin\alpha)t$. The object is at its maximum height when $v = 0$ and in that case the time is $t = \dfrac{v_0}{g\sin\alpha}$. To return to its initial position it take twice as much time, so the total time is $t = \dfrac{2v_0}{g\sin\alpha}$.

50. Alin and Carlos are in empty space (where there is no gravity and their gravitational attraction is so small that it can be considered to be zero) separated by a distance $\mathbf{d}$. Alin throws a ball that reaches Carlos at time $\mathbf{T}$. Denote by $\mathbf{m_A}, \mathbf{m_C}$, and $\mathbf{m_B}$ the masses of Alin, Carlos and the ball respectively. Find Alin's speed after the ball is thrown.

(A) $\dfrac{\mathbf{m_A T}}{\mathbf{m_B d}}$ (B) $\dfrac{\mathbf{m_B d}}{\mathbf{m_A T}}$ (C) $\dfrac{\mathbf{m_B d}}{(\mathbf{m_A} + \mathbf{m_C})\mathbf{T}}$

(D) $\dfrac{(\mathbf{m_A} + \mathbf{m_C})\mathbf{T}}{\mathbf{m_B d}}$ (E) $\mathbf{0}$

Answer: B

Denote Alin's speed after the ball is thrown by $\mathbf{s_A}$. The speed of the ball is $\dfrac{\mathbf{d}}{\mathbf{T}}$. The linear momentum $\mathbf{p}$ of an object is defined as the product of the object's mass $\mathbf{m}$ and its velocity $\mathbf{v}$. The absolute value of the momentum of the ball is $|\mathbf{p}_{ball}| = \dfrac{\mathbf{m_B d}}{\mathbf{T}}$. The absolute value of Alin's momentum is $|\mathbf{p}_{Alin}| = \mathbf{m_A s_A}$. By the conservation law of the linear momentum we get $\dfrac{\mathbf{m_B d}}{\mathbf{T}} = \mathbf{s_A m_A}$, so $\mathbf{s_A} = \dfrac{\mathbf{m_B d}}{\mathbf{m_A T}}$.