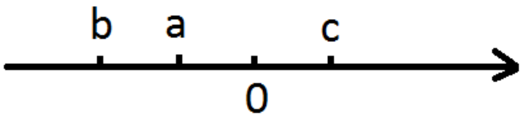
Math Tournament 2016 - Ciphering

Round 1

1. Let a, b and c be points equally spaced in the number line as shown below.



   Simplify the expression

$$\sqrt{a^2} - |a+b| + \sqrt{(c-a)^2} + |b+c|.$$

   Answer : $-a$.

   It is easy to see $c > 0 > a > b$, therefore

$$\sqrt{a^2} - |a+b| + \sqrt{(c-a)^2} + |b+c| = -a - (-(a+b)) + c - a - (b+c).$$

2. If $a = \sqrt{2} + 1$, what is the value of $1 + \dfrac{1}{2 + \frac{1}{a}}$?

   Answer: $\sqrt{2}$.

$$1 + \frac{1}{2 + \frac{1}{a}} = 1 + \frac{1}{2 + \frac{1}{\sqrt{2}+1}} = 1 + \frac{1}{2 + \sqrt{2} - 1} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \sqrt{2} - 1 = \sqrt{2}.$$

3. One third of the number 27^{81} can be written in the form 9^A where A is a three digit number. Find the value of A.

   Answer : 121.

   We have

$$\frac{1}{3} \cdot 27^{81} = \frac{1}{3} \cdot 3^{243} = 3^{242} = 9^{121}.$$

4. If $2x^2 + y^2 - 2xy - 4x + 4 = 0$, what is the value of $x + y$?

   Answer : 4.

   Rearranging the original equation, we have

$$\begin{aligned} 2x^2 + y^2 - 2xy - 4x + 4 &= x^2 + y^2 - 2xy + x^2 - 4x + 4 \\ &= (x-y)^2 + (x-2)^2 = 0. \end{aligned}$$

   Therefore, $x = y = 2$.

5. Let a and b be real numbers such that $a \neq b$, $a^2 + 3a = 2$, and $b^2 + 3b = 2$. Find the value of $(1 + a)(1 + b)$.

   Answer : -4.
   It is easy to see a and b are the roots of $x^2 + 3x - 2 = 0$. Therefore, $a + b = -3$, $ab = -2$, and

$$(1 + a)(1 + b) = 1 + a + b + ab = 1 - 3 - 2 = -4.$$

6. Evaluate $\log_2 2\sqrt{4\sqrt{2}} - \dfrac{1}{4}$.

   Answer : 2.

   We have

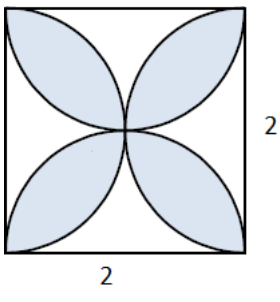
$$\log_2 2\sqrt{4\sqrt{2}} - \frac{1}{4} = \log_2 4\sqrt{2^{\frac{1}{2}}} - \frac{1}{4} = \log_2 2^{\frac{9}{4}} - \frac{1}{4} = \frac{9}{4} - \frac{1}{4} = 2.$$

7. The sum of two prime numbers is 99. What is the product of these two prime numbers?

   Answer : 194.

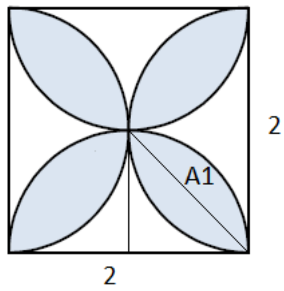
   Suppose a and b are prime numbers such that $a + b = 99$, which means either a or b will be an even number. There is only one even prime number, 2. The answer is 194.

8. The figure shows a square and four semicircles generated with each side of the square as a diameter. If the side length of the square is 2, find the area of the shaded region.



   Answer : $2\pi - 4$.

The area of A_1 can be obtained by deducting the area of the triangle from the area of the quarter circle with radius 1. We have $A_1 = \dfrac{\pi}{4} - \dfrac{1}{2}$. All shaded area$= 8 \cdot A_1 = 2\pi - 4$.



Round 2

1. How many digits are in the base-ten numeral $16^{11} \cdot 5^{40}$?

   Answer : 42

   We have

   $$16^{11} \cdot 5^{40} = 2^{44} \cdot 5^{40} = 2^4 \cdot 10^{40} = 16 \cdot 10^{40}.$$

   There are 42 digits.

2. Amber, Ben, and Cathy shared a pizza. Amber ate $\frac{1}{5}$ of the pizza, Ben ate one-half as much as Cathy did. Find how much of the pizza Ben ate.

   Answer : $\frac{4}{15}$.

   Since Amber ate $\frac{1}{5}$, the remaining $\frac{4}{5}$ is shared by Ben and Cathy. Assume Cathy ate x portion of the pizza. Then Ben should ate $\frac{1}{2}x$. Therefore $x + \frac{1}{2}x = \frac{4}{5}$. Solving for x gives $x = \frac{8}{15}$. Ben ate $\frac{1}{2}x = \frac{4}{15}$ of the pizza.

3. Let a and b be real numbers such that $a + b = 1$ and $a^2 + b^2 = 4$. What is the value of $\frac{b^2}{2+a} + \frac{a^2}{2+b}$?

   Answer : 3.

   Since $b^2 = 4 - a^2$ and $a^2 = 4 - b^2$, we can simplify

   $$\frac{b^2}{2+a} + \frac{a^2}{2+b} = \frac{4-a^2}{2+a} + \frac{4-b^4}{2+b} = 2 - b + 2 - a = 4 - (a+b) = 4 - 1 = 3.$$

4. Let a be a real number such that $a > 0$ and $|(a + 2i)(1 + i)| = 4$. Find the value of a.

   Answer : 2.

   Rewrite

   $$|(a + 2i)(1 + i)| = \sqrt{a^2 + 4} \cdot \sqrt{1 + 1} = 4 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2.$$

5. In a barn with chickens and rabbits, there are 7 heads and 20 legs. How many chickens are there? (A chicken has 2 legs and a rabbit has 4 legs.)

   Answer : 4.

   Let x denote the number of chicken and y denote the number of rabbits. Then $x+y = 7$ and $2x + 4y = 20$. We have $x = 4$ and $y = 3$.

6. Let $f(x) = (2 - \sin\sqrt{x})^2$. What is the maximum value of $f(x)$?

   Answer : 9.

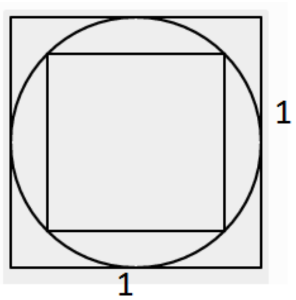
   Since $|\sin\sqrt{x}| \le 1$, then

   $$(2 - \sin\sqrt{x})^2 \le (2 + 1)^2.$$

7. Let $M = \{1,2,3\}$ and $N = \{2a-1|a \in M\}$. Find $M \bigcap N$.

   Answer : $\{1{,}3\}$.
   Note that $N = \{2a-1|a \in M\} = \{1,3,5\}$. The solution is $M \bigcap N = \{1,3\}$.

8. If the area of the outer square is 1, what is the area of the small square inside?



   Answer : $\frac{1}{2}$.

   The diameter of the circle is 1. The diagonal line of the inner square is 1 too. The side of the inner square is $\frac{\sqrt{2}}{2}$. Therefore, the area of the inner square is $\frac{1}{2}$.