Math Tournament 2016 - Ciphering

Round 1

1. The lines $y - 2x - 4 = 0$ and $2y - ax + 4 = 0$ are parallel. Find the value of a.

   Answer : $a = 4$.

   The slopes of these lines are 2 and $\frac{a}{2}$. From $2 = \frac{a}{2}$, we must have $a = 4$.

2. If x is a real number such that $|2015 - x| + \sqrt{x - 2016} = x$, find the value of $x - 2015^2$.

   Answer : 2016.

   Since $x \geqslant 2016$, and

$$|2015 - x| + \sqrt{x - 2016} = x \Rightarrow x - 2015 + \sqrt{x - 2016} = x \Rightarrow \sqrt{x - 2016} = 2015$$
$$x - 2015^2 = 2016.$$

3. Suppose a and b are the roots of $x^2 - 3\sqrt{3}x + \sqrt{3} = 0$. Find the value of $\frac{1}{a} + \frac{1}{b}$.

   Answer: 3.

   By the Vieta relation, we have $a + b = 3\sqrt{3}, ab = \sqrt{3}$.

$$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab} = 3.$$

4. Find the largest value of c such that $x^2 + cx + 1 \geq 0$ for all real numbers x.

   Answer : 2.

   Since the quadratic $x^2 + cx + 1 \geq 0$, we must have

$$\triangle = c^2 - 4 \leq 0 \Rightarrow -2 \leq c \leq 2$$

5. If the function $f(x) = x - \frac{1}{x}$ is defined on $[1, 2]$, find the minimum value of $f(x)$.

   Answer : 0.

   Both function $G(x) = x$ and $F(x) = -\frac{1}{x}$ are monotone increasing on $[1, 2]$, the minimum of $f(x)$ is $f(1) = G(1) + F(1) = 0$.

6. Write the expression $\log_2 2\sqrt{2\sqrt{2}} - \frac{3}{4}$ as an integer without using radicals and exponents.

   Answer : 1.

   Since

$$\log_2 2\sqrt{2\sqrt{2}} - \frac{3}{4} = \log_2 2\sqrt{2 \cdot 2^{\frac{1}{2}}} - \frac{3}{4} = \log_2 2^{\frac{7}{4}} - \frac{3}{4} = \frac{7}{4} - \frac{3}{4} = 1.$$

7. Let a and b be real numbers such that $a > b > 0$ and $a^2 + b^2 = 2016ab$. Find the simplified value of $\dfrac{a+b}{a-b}$.

Answer : $\sqrt{\frac{1009}{1007}}$.

Since $a^2 + b^2 + 2ab = 2018ab$ and $a^2 + b^2 - 2ab = 2014ab$

$$\frac{a+b}{a-b} = \sqrt{\frac{(a+b)^2}{(a-b)^2}} = \sqrt{\frac{2018ab}{2014ab}} = \sqrt{\frac{1009}{1007}}$$

8. For the function $f(x) = \dfrac{1}{x}$, find the expression of $f(f(f(x)))$.

Answer : $\frac{1}{x}$.

$$f(f(x)) = \frac{1}{\frac{1}{x}} = x, \quad f(f(f(x))) = \frac{1}{x}.$$

Round 2

1. Let x be a real number such that $|x-2016|+\sqrt{2016-x}=2$. Find the value of $x+1$.

   Answer : 2016.

   Clearly, $2016 \geqslant x$. And

$$\begin{aligned}|x-2016|+\sqrt{2016-x}=2 \Rightarrow 2016-x+\sqrt{2016-x}-2=0 \\ \Rightarrow (\sqrt{2016-x}-1)(\sqrt{2016-x}+2)=0 \\ \Rightarrow \sqrt{2016-x}=1.\end{aligned}$$

2. Suppose a and b are the roots of $x^2-3x+1=0$ and $a>b>0$. Find the value of $\frac{1}{b}-\frac{1}{a}$.

   Answer : $\sqrt{5}$.

   By the Vieta relation, we have

$$a+b=3, ab=1 \Rightarrow (a-b)^2=(a+b)^2-4ab=5.$$

   Therefore

$$\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}=\frac{\sqrt{(a-b)^2}}{ab}=\sqrt{5}$$

3. The numbers a_1, a_2, a_3, a_4, and a_5 are positive consecutive integers whose sum is 510. What is a_5?

   Answer : 104.

   $a_l+a_2+\cdots+a_5=a_1+(a_1+1)+(a_1+2)+(a_1+3)+(a_1+4)=5a_1+10=510$, which gives $a_1=100$. So $a_5=a_1+4=104$.

4. The function $f(x)=\dfrac{1}{x^2+2\sqrt{c}x+c+1}$ is defined for all real numbers x. Find all possible values of c.

   Answer : $[0,+\infty)$.

   Note that $\sqrt{c}$ is a real number only for c in $[0,+\infty)$. The quadratic polynomial $x^2+2\sqrt{c}x+c+1\neq 0$, for all c in $[0,+\infty)$ since the discriminant

$$\triangle = 4c-4c-4<0.$$

5. For all positive integers n, let $f(n)=\log_{2016} n^{2016}$. Find the value of $f(32)+f(9)+f(7)$. (**Note:** $2016=2^5 3^2 7$.)

   Answer : 2016

$$\begin{aligned}&\log_{2016} 32^{2016}+\log_{2016} 9^{2016}+\log_{2016} 7^{2016} \\ &=\log_{2016}(32\cdot 9\cdot 7)^{2016}=\log_{2016}(2016)^{2016}=2016\end{aligned}$$

6. The first two terms of a geometric sequence are $\sqrt[9]{2^7}$ and $\sqrt[3]{4}$. What is the forth term of the sequence?

   Answer : $2^{\frac{4}{9}}$.

   Noting that $\sqrt[9]{2^7} = 2^{\frac{7}{9}}$ and $\sqrt[3]{4} = 2^{\frac{2}{3}} = 2^{\frac{6}{9}}$, the third term is $2^{\frac{5}{9}}$ and the forth term is $2^{\frac{4}{9}}$.

7. Find the value of $\log_2 2\sqrt{2\sqrt{2}} - \ln e\sqrt{e\sqrt{e}}$.

   Answer : 0.

   Note that

$$\log_2 2\sqrt{2\sqrt{2}} = \log_2 2\sqrt{2 \cdot 2^{\frac{1}{2}}} = \log_2 2^{\frac{7}{4}} = \frac{7}{4}$$
$$\ln e\sqrt{e\sqrt{e}} = \log_e e\sqrt{e \cdot e^{\frac{1}{2}}} = \log_e e^{\frac{7}{4}} = \frac{7}{4}$$

8. If the area of circle A is 64% of the area of circle B, then the radius of circle B is what percentage of the radius of circle A?

   Answer : 125%.

   Let r_1 be the radius of circle A and r_2 be the radius of circle B. Then $\frac{\pi r_1^2}{\pi r_2^2} = 0.64$. So $r_1/r_2 = 0.8$ and $r_2/r_1 = 1.25$.