Forty-eight Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our virtual campus. We wish you success on this test and in your future studies.

Instructions

This is a 240-minute, 30-problem, multiple choice examination. There are six possible responses to each question. You should select the one "*best*" answer for each problem.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Important notation and facts to consider on this test

All numbers when referred to their digits are considered written in base 10 unless otherwise stated.

- $2022 = 2 \times 1011 = 2 \times 3 \times 337$ (prime factorization), $337 = 16^2 + 9^2$
- Given two non-negative integers, a and b, $\gcd(a, b)$ denotes the greatest positive integer which divides both a and b. Example: $\gcd(2022, 672) = 6$.
- When $\gcd(a, b) = 1$ (see the above definition), we say that a and b are relatively prime or co-prime. Example: 7 and 2022 are relatively prime (see the prime factorization of 2022).
- For integers a and b, and n a natural number, $a \equiv b \pmod{n}$ simply means n divides $a - b$. For exmple, $3^6 \equiv 1 \pmod{7}$ since 7 divides

$$3^6 - 1 = 27^2 - 1 = 26 \cdot 28 = 7 \cdot 4 \cdot 26.$$

- $[XYZW...]$ means the area of the polygon $XYZW...$

- **Corollary to Ceva's Theorem:** Given triangle ABC and three Cevians $\overline{AP}$, $\overline{BN}$, and $\overline{CM}$, concurrent at O (see Figure 1), then

$$\frac{OA}{OP} = \frac{MA}{MB} + \frac{NA}{NC}.$$

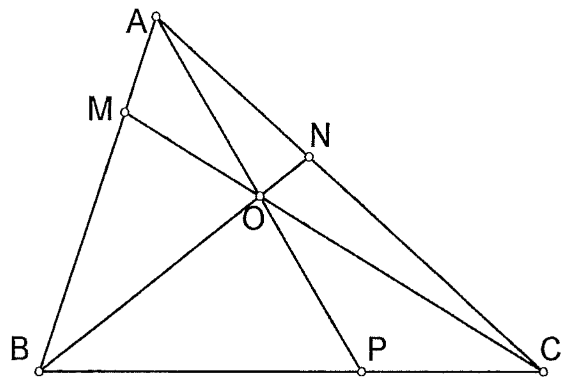


Figure 1, Ceva's Corollary

- Some useful Taylor expansions:

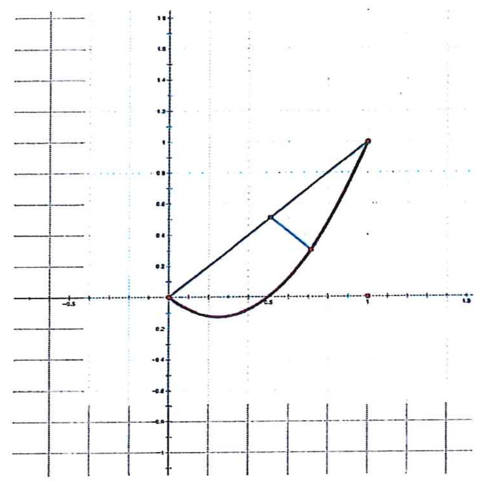
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad x \in (-1, 1],$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad x \in \mathbb{R},$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad x \in [-1, 1],$$

Problem 1

- In the xy-plane, consider the region between $y = 2x^2 - x$ and $y = x$. Rotate this region around the line of equation $y = x$. Calculate the volume of the solid obtained and write it as $\pi\sqrt{2}\frac{m}{n}$, with the fraction $\frac{m}{n}$ in reduced form. What is $n-m$?



(A) 10 (B) 11 (C) 12

(D) 13 (E) 14 (F) 15

Problem 2

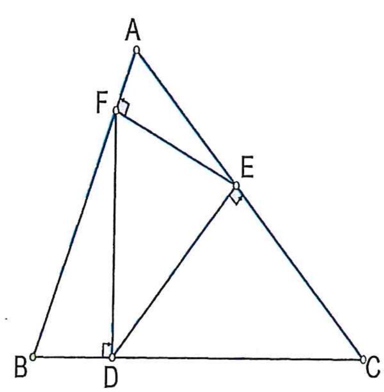
- Let A be the set of all integers n, $1 \leq n \leq 2021$, which are relatively prime with 2022. Find the sum of the digits (base 10) of the number N defined below

$$N = \frac{\sum_{n \in A} n}{2022}.$$

(A) 15 (B) 14 (C) 13 (D) 12 (E) 11 (F) 10

Problem 3

- Let ABC be an isosceles triangle with sides $AB = AC = 5$ and $BC = 6$. Suppose D, E and F are points on sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively, such that $\overline{FD}$ is perpendicular to $\overline{BC}$, $\overline{DE}$ is perpendicular to $\overline{CA}$, and $\overline{EF}$ is perpendicular to $\overline{AB}$ (see the figure on the right). The segment FA is equal to a fraction which is, in reduced form, equal to $\frac{m}{n}$. What is $n - m$?



(A) 10 (B) 9 (C) 8

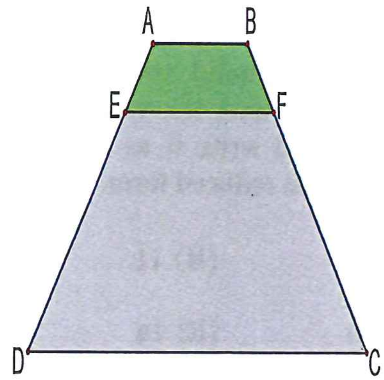
(D) 6 (E) 7 (F) 5

Problem 4

- In the accompanying figure, $ABCD$, $ABFE$ and $EFCD$ are trapezoids with $\overline{AB}$, $\overline{EF}$ and $\overline{CD}$ parallel. We let $x = EF$, such that $\frac{[EFCD]}{[ABFE]} = 7$ (ratio of areas). Knowing that $AB = a < CD = b$ are the solutions in integers of the Diophantine equation

$$3(a^2 + b^2) = 2022,$$

and that x is an integer, what is the sum of the digits of x?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5 (F) 6

Problem 5

- For any real number $x \in (0,1)$, we define

$$f(x) = \lim_{r \to 0} \left(\frac{(1-x)^{r+1} + (1+x)^{r+1}}{2} \right)^{\frac{1}{r}}.$$

The following limit

$$\lim_{x \to 0} f(x)^{\frac{1}{x^2}}$$

exists and it is equal to $e^{\frac{m}{n}}$, for some relatively prime positive integers m and n. What is $n - m$?

(A) 0 (B) -1 (C) 1 (D) 2 (E) -2 (F) 3

Problem 6

- For any real number $x \in \mathbb{R}$, we have the identity

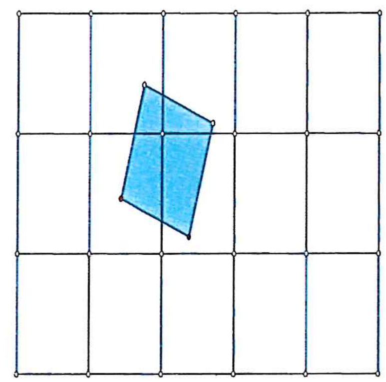
$$(\sin x + \cos x - 1)^2 = a(b - \sin x)(c - \cos x).$$

What is $a + b + c$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5

Problem 7

- Suppose that the plane is tiled with an infinite checkerboard of unit squares. Another square, with sides equal to $r \in [\frac{1}{\sqrt{2}}, 1]$, is dropped on the plane at random with position and orientation independent of the checkerboard tiling. The probability that it does not cover any of the corners of the squares of the checkerboard is equal to



$$P = \frac{4}{\pi}\Big[\frac{\pi}{4}(1-mr^2) + 2r^2\arcsin(r) + nr\sqrt{1-r^2} - r^2 - \frac{1}{2}\Big],$$

for two positive integers m and n. What is $m + n$?

(A) 7 (B) 6 (C) 5

(D) 4 (E) 3 (F) 2 .

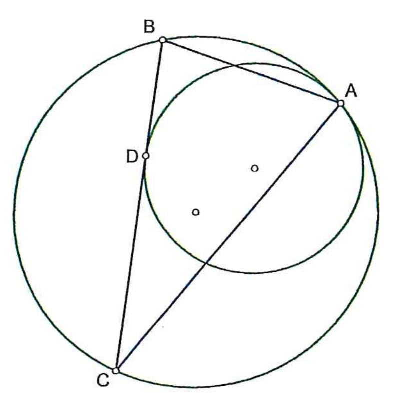
Problem 8

- A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. The smallest number of hops needed for the grasshopper to reach the point $(2022, 2022)$ is N. What is the sum of the digits of N?

(A) 20 (B) 22 (C) 13 (D) 21 (E) 14 (F) 25

Problem 9

- Let c_1 and c_2 be two circles, with c_2 tangent to, and inside of c_1, at the point A. We take a point D on c_2 and construct a tangent chord $\overline{BC}$ to c_2 with the endpoints B and C on c_1 (see the Figure on the right). The sides of the triangle ABC are $AB = 5$, $AC = BC = 20$. What is the length of AD?



(A) 7 (B) 6 (C) 5

(D) 4 (E) 3 (F) 2

Problem 10

- The following limit

$$L := \lim_{n\to\infty} \int_0^{1/2} \frac{(1+f_{n-1}(x))^{n+1}}{(1+f_n(x))^n} dx,$$

where $f_n(x) = x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}$, $x \in [0,1]$, is given by $L = m - n\ln 2$ for two rational numbers m and n. What is $\frac{m}{n}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5 (F) 6