Forty-seventh Annual Columbus State Invitational Mathematics Tournament

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Important notation and facts to consider on this test

All numbers when referred to their digits are considered written in base 10 unless otherwise stated.

- $2021 = 2025 - 4 = 45^2 - 2^2 = 47 \times 43$ (prime factorization)
- $[XYZW...]$ means the area of the polygon $XYZW...$
- **Corollary to Ceva's Theorem:** Given triangle ABC and three Cevians $\overline{AP}$, $\overline{BN}$, and $\overline{CM}$, concurrent at O (see Figure 1), then

$$\frac{OA}{OP} = \frac{MA}{MB} + \frac{NA}{NC}.$$

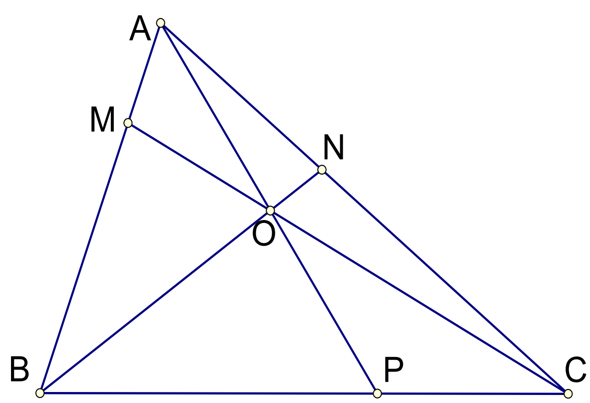
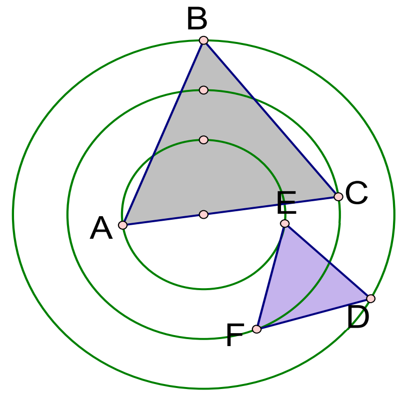


Figure 1, Ceva's Corollary

1. Given three concentric circles of radii a, b and c, there exists an equilateral triangle of side lengths equal to x and with vertices on these circles, as in the adjacent figure, if and only if



$$x^4 + a^4 + b^4 + c^4 - x^2a^2 - x^2b^2 = x^2c^2 + a^2b^2 + b^2c^2 + c^2a^2. \tag{1}$$

If $a = 3$, $b = 5$ and $c = 7$, there are two solutions for x. One is $x = 8$ and the other is given by the equation $x^2 = d$. What is the value of d?

(A) 17 (B) 18 (C) $\boxed{19}$

(D) 20 (E) 21

Solution: (Method I) Substituting the given values of a, b and c in (1) we get $0 = 1216 - 83x^2 + x^4$ and after the given factor is used we get $0 = (x^2 - 64)(x^2 - 19)$. Hence the answer is C.

(Method II) The equation is biquadratic and so it factors as $(x^2 - 64)(x^2 - d) = 0$ this means $64 + d = a^2 + b^2 + c^2 = 9 + 25 + 49$ and so $d = 83 - 64 = 19$. ■

2. Find the smallest natural number n such that there exists a choice of the signs that makes the following equality true

$$\pm 1 \pm 2 \pm 3 + \cdots \pm n = 2021.$$

(A) 63 (B) 64 (C) $\boxed{65}$ (D) 69 (E) 73

Solution: Answer is C. Since $\sum_{k=1}^{63} = \frac{63 \cdot 64}{2} = 2016 < 2021$, $n = 63$ is not going to cut it. For $n = 64$ the corresponding sum, independent of the choice of signs, it is going to be an even number and so it cannot be 2021. For $n = 65$ we have

$$2021 = \left(\sum_{k=1}^{65} k \right) - 2(61 + 1).$$

■

3. Suppose that for three real numbers x, y and z, we have $x+y+z=3$, $x^2+y^2+z^2=19$, and $x^3+y^3+z^3=84$. What is the value of xyz?

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: Since $9=(x+y+z)^2=19+2(xy+yz+xz)$ it follows that $xy+yz+xz=-5$. Another identity that we can use here is

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz) \implies$$
$$3xyz=84-3(19+5)=3(28-24)=3(4).$$

Therefore, we have $xyz=4$, and so the correct answer is D. ■

4. The function f is real valued defined on the whole real line and satisfies $f(f(x))=\frac{3^x-1}{2}$ for all x. Knowing that f is (strictly) decreasing, find $f(1)$.

(A) $\boxed{0}$ (B) 1 (C) 3 (D) 1/3 (E) -1

Solution: We observe that f is one-to-one, and so it must be strictly decreasing. Let us consider the function defined on the whole real line given by $g(x)=\frac{3^x-1}{2}$ for all x. We notice that $g''(x)=\frac{1}{2}3^x(\ln 3)^2>0$ for all x. Hence, g is a convex function. We observe that if $f(1)=a$ then $f(a)=f(f(1))=g(1)=1$. Also, we have $g(0)=0$. Becasue g is a convex function the equation $g(x)=x$ must have at most two solutions. In other words, 0 and 1 are the only solutions of the equation $g(x)=x$. On the other hand, $g(a)=f(f(a))=f(1)=a$ and so a must be 1 or 0. Let's say $a=1$. Then $f(0)=b$ implies $f(b)=f(f(0))=0$ and so $g(b)=f(0)=b$. Therefore, b is 0 or 1. We cannot have $b=1$ since f is one-to-one. Then $b=0$ and again this is in contradiction with f being decreasing ($f(0)>f(1)$). It remains that $a=0$ and the asnwer is A. ■

5. A set S is built with only applying any of the following rules:

(a) $2 \in S$

(b) if $t \in S$ then $t+5$ is also in S

(c) if $t \in S$ then $3t$ is also in S.

The number time the rules are applied is not limited and it is irrelevant. What is the greatest number n, $n \leq 2021$, which is not in S?

(A) 2021 (B) $\boxed{2020}$ (C) 2019 (D) 2018 (E) 2017

Solution: The number 2021 is in S, since we can use the rules to get $7=2+5\in S$, $21=7\cdot 3\in S$, $26=21+5\in S$ and $31=26+5\in S$, ..., $2016+5=2021\in S$.

However, the number 2020 can only be in S if 2015 was in S. Then this is predicated by $2010 \in S$. At this point 2010 is divisible by 3, so we may have $2010/3 = 670 \in S$. In any case, what rules we use to go back the numbers on the way are always ending in 0 or 5. As a result, the resulting sequences canont be traced back to 2. So, 2020 cannot be in S. Therefore, the answer is B. ■

6. The set $A = \{1, 48, 95, ...\}$ is the arithmetic progression of common difference of 47 and initial term 1, and the set $B = \{2, 45, 88, ...\}$ is the arithmetic progression of common difference of 43 and initial term 2. If the elements of the intersection $A \cap B$ are listed in non-decreasing order, $A \cap B = \{a_1, a_2, a_3, ...\}$, find the sum of the digits of a_2.

   (A) 15 (B) 16 (C) 17 (D) 18 (E) $\boxed{19}$

Solution: We have the sequence in A given by $a_n = 1 + 47(n-1)$, $n = 1, 2, ...$ and the sequence in B given by the formula $b_m = 2 + 43(m-1)$. The common elements can be written as

$$1 + 47(n-1) = 2 + 43(m-1) \Leftrightarrow 47(n-1) - 43(m-1) = 1.$$

The key observation here is $47 = 43 + 4$ and so we have

$$43(n-1) + 4(n-1) - 43(m-1) = 1 \Leftrightarrow 43(n-m) + 4(n-1) = 1,$$

which has a simple solution $n - m = -1$ and $n - 1 = 11$ (or $n = 12$). Thus $m = n + 1 = 13$, which give the first term in the intersection $A \cap B$, $a_{12} = 1 + 47(11) = 518$ and $b_{13} = 2 + 43(12) = 518$. Therefore, the elements in the intersection $A \cap B$ are of the form $a_n = 518 + 47(n-1) = 518 + 43(m-1)$ with $n = 12, ...$ and $m = 13, ...$ In fact we see that $n - 1 = 43(k-1)$ and so the common elements are of the form $c_k = 518 + 2021(k-1)$, $k = 1, 2, 3, ...$ So, $c_2 = 2539$ which gives the answer to this problem

$$sd(c_2) = sd(2539) = 2 + 5 + 3 + 9 = \boxed{19}. \quad \square$$

7. Two real numbers x and y satisfy

$$x^2 + xy + y^2 - x + y + 1 = 0.$$

   What is the value of $2x + y$?

   (A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: The equation can be written as

$$(x + y)^2 + (x - 1)^2 + (y + 1)^2 = 0.$$

The only real numbers satisfying this are $x = 1$ and $y = -1$. This implies that $2x + y = 1$. Hence we have A as the correct answer. ■

8. A fly starts at the origin (of the coordinates plane) and goes 1 unit up, $\frac{1}{2}$ unit right, $\frac{1}{4}$ unit down, $\frac{1}{8}$ unit left, $\frac{1}{16}$ unit up, etc., all the way to infinity. Let $P = (\frac{m}{n}, \frac{p}{n})$ be the coordinates of the point where it ends up (the two coordinates are rational numbers written in reduced form). What is $m + p - n$?

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: Let us write the sequence of positions as a sum of multiples of the basis of vectors $u =< 1, 0 >$ and $v =< 0, 1 >$. So, at the beginning we have v, and then $v + \frac{1}{2}u$, followed by $v + \frac{1}{2}u - \frac{1}{4}v$. We can see the point P is the result of an infinite sum that can be written as

$$v + \frac{1}{2}u - \frac{1}{4}v - \frac{1}{8}u + \frac{1}{16}v + \frac{1}{32}u - \frac{1}{64}v - \frac{1}{128}u + ... \text{ or}$$

$$(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + ...)v + (\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{128} + ...)u.$$

But the two infinite sums are classical geometric series that can be calculated with the formula

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}, \quad |r| < 1.$$

In this cases, $r = -\frac{1}{4}$, and then we can continue

$$\vec{OP} = \frac{1}{1 - (-\frac{1}{4})}v + \frac{1}{2} \cdot \frac{1}{1 - (-\frac{1}{4})}u = \frac{4}{5}v + \frac{2}{5}u.$$

Hence, we have $m = 2$, $n = 5$ and $p = 4$. This gives the answer A. ■

9. The cubic equation $x^3 - 36x^2 + ax - 1428 = 0$ has three solutions in arithmetic progression. Find the sum of the digits of a.

(A) 10 (B) $\boxed{11}$ (C) 12 (D) 13 (E) 14

Solution: We can denote the solutions as $x_1 = a - t$, $x_2 = a$, and $x_3 = a + t$. Using Viète′s Relations we see that $x_1 + x_2 + x_3 = 36$ or $3a = 36$ which gives $a = 12$. Also, we have $x_1x_2x_3 = 1428$ or $(144 - t^2)12 = 1428$. Solving for t we obtain $t = \pm 5$. Then the coefficient a is obtained by

$$(x_1 + x_3)x_2 + x_1x_3 = 2a^2 + a^2 - t^2 = 3a^2 - 25 = 407,$$

which gives the answer B. ■

10. The number $abcdefg$ is formed with only the digits 1 and 2 (in base 10) and it is divisible by 128. What is $a+b+c+d+e+f+g$?

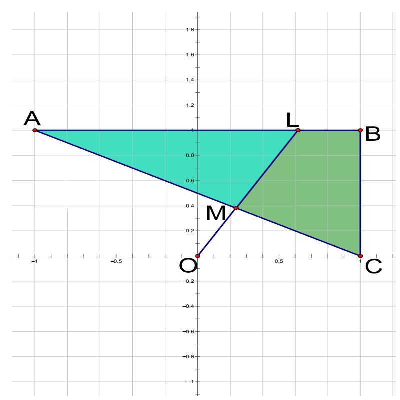
(A) 10 (B) $\boxed{11}$ (C) 12 (D) 13 (E) 14

Solution: Let us denote the number $abcdefg$ by N. First we observe that g must be equal to 2 since N must be even. Then $f2 = N - 100abcde$ is divisible by 2^2 which forces $f = 1$. Next, we have

$$e12 = N - 1000abcd \;\text{ divisible by }\; 2^3.$$

This gives $e = 1$. We can continue this way until we determine all the digits of N: $N = 2122112$. Therefore, the sum of the digits of N is 11. ■

11. In the accompaning figure the coordinates of the triangle ABC are $A = (-1, 1)$, $B = (1, 1)$ and $C = (1, 0)$. We denote the slope m of the line $\overline{LM}$ which passes through the origin of the axis, the point O, and divides the area of the triangle ABC in half. The equation that m satisfies is the quadratic $x^2 - x - \alpha = 0$. What is the value of α?



(A) $\boxed{1}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$

(D) 2 (E) $\frac{5}{4}$

Solution: First let us observe that $[ABC] = \frac{AB \cdot BC}{2} = 1$. The equation of the line $O - M - L$ is $y = mx$, and the equation of $A - M - C$ is $y = \frac{1-x}{2}$. It is easy to see that L has coordinates $(\frac{1}{m}, 1)$.

The point M has the x-coordinate the solution of $mx = \frac{1-x}{2}$ or $x_M = \frac{1}{2m+1}$. Then its y-coordinate is $y_M = mx_M = \frac{m}{2m+1}$. We can calculate the height of the triangle ALM corresponding to the base AL and vertex M as $1 - y_M = \frac{m+1}{2m+1}$. So, the area of AML gives the equation in m:

$$[AML] = \frac{1}{2} \implies (1 + \frac{1}{m})\frac{m+1}{2m+1} = 1 \implies$$
$$(m+1)^2 = 2m^2 + m \implies m^2 - m - 1 = 0.$$

Therefore, the correct answer is A. ■

Remark: In fact, m is the Golden ratio.

12. How many pairs of positive integers (x, y) satisfy $x^2 - y^2 = 2021^2$?

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: The equation can be written as $(x-y)(x+y) = 43^2 \cdot 47^2$. The number of divisors of 2021^2 is then $(2+1)(2+1) = 9$. So, with the exception of 2021 we can group them in 4 pairs $(d, 2021^2/d)$. Each such pair gives a solution for (x, y) since the system $x - y = d$ and $x + y = 2021^2/d$ is going to have positive integer solutions. Therefore, the number of solutions is 4 and the answer is D. ■

13. Rolling four regular dice (unbiased, and faces numbered from 1 to 6), the probability to cast a sum of 15 is equal to $\mathcal{P} = \frac{m}{n}$ for some positive integers m and n. Assuming the fraction $\frac{m}{n}$ is written in reduced form, what is $n - m$?

(A) 234 (B) 267 (C) 212 (D) 256 (E) $\boxed{289}$

Solution: The number of all possible events is $6^4 = 1296$. The number of favorable events is equal to the coefficient of x^{15} in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^4$. This is the same as the coefficient of x^{11} in the expansion of $(1+x+x^2+x^3+x^4+x^5)^4$. It is not hard to see that $(1+x+x^2+x^3+x^4+x^5)^2$ is equal to

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}.$$

So, the coefficient of x^{11} in the expansion of $(1+x+x^2+x^3+x^4+x^5)^4$ is equal to

$$2(2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5) = 140.$$

This implies that $\mathcal{P} = \frac{140}{1296} = \frac{35}{324}$ which gives $n - m = 324 - 35 = 289$. ■

14. It is known that every positive integer can be written as a sum of non-consecutive Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$) in an unique way. Taking into consideration this writing for 2021,

$$2021 = F_{n_1} + F_{n_2} + \cdots + F_{n_k},$$

find k.

(A) 5 (B) $\boxed{4}$ (C) 3 (D) 2 (E) 1

Solution: Answer B. One can check that

$$2021 = F_{17} + F_{14} + F_9 + F_7.$$

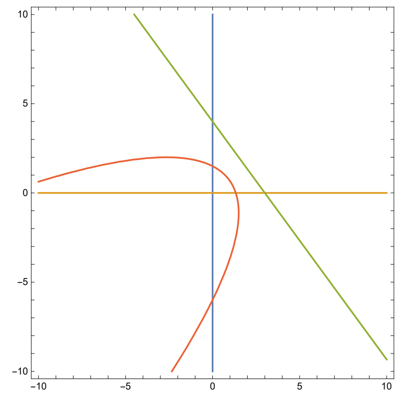
15. The equation of the parabola (see the red curve in the figure on the right) with focus $(0,0)$ and directrix the line of equation $\frac{x}{3}+\frac{y}{4}=1$ is

$$(3x-4y)^2+a(4x+3y)=b^2.$$

What is the value of $\frac{a}{b}$?

(A) 1 (B) $\boxed{2}$ (C) 3

(D) 4 (E) 5



Solution: The equation of the parabola is given by the property that it is the locus of the points equally distant to the focus and the directrix. Hence the equation is

$$\sqrt{x^2+y^2}=\frac{|\frac{x}{3}+\frac{y}{4}-1|}{\sqrt{\frac{1}{3^2}+\frac{1}{4^2}}} \Leftrightarrow 25(x^2+y^2)=(4x+3y-12)^2 \Leftrightarrow$$

$$(3x-4y)^2+24(4x+3y)=144=12^2.$$

Hence, we have $a/b=2$, and so the correct answer is B. ■

16. The sum of two real numbers is n and the sum of their squares is $n+2021$, for some positive integer n. What is the maximum possible value of n?

(A) $\boxed{64}$ (B) 66 (C) 63 (D) 67 (E) 62

Solution: If we denote by x and y those two real numbers, we have the system $x+y=n$ and $x^2+y^2=n+2021$. We need to have $2(x^2+y^2)-(x+y)^2=(x-y)^2\geq 0$ so $2(n+2021)-n^2\geq 0$. This last inequality is equivalent to $(n-1)^2\leq 4043$ or $n\leq 1+\sqrt{4043}\approx 64.5846$. So, we may try $n=64$ and we get indeed two real solutions of the system:

$$x=32+\frac{\sqrt{74}}{2} \quad \text{and} \quad y=32-\frac{\sqrt{74}}{2}.$$

Hence, the answer is A. ■

17. A box has 6 red balls and 3 blue balls. Bill picks one ball at the time, from the box, without replacement, until all of balls of the same color are out. If $\frac{p}{q}$ is the probability that the last ball Bill picks is blue, written as a reduced fraction, what is $q-p$?

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We can think of this experiment as choosing 8 balls laid out in order. One ball will be left. If what is left is a red ball, this is then a favorable event since we had picked all of the blue ones within the 8 ones. If what is left is a blue ball, we had picked all the red ones first, and so this is an unfavorable event. We obtain $\binom{8}{3} = \frac{8(7)(6)}{1(2)(3)} = 56$ possibilities for the first case and $\binom{8}{6} = \binom{8}{2} = 28$ arrangements for the second case. Therefore, the probability is $\frac{56}{56+28} = \frac{2}{3}$. So, the answer is A.

■

18. For two positive integers a and b the following limit exists

$$L := \lim_{x \to 0} \frac{x^2}{4 - \sqrt[3]{a+bx} - \sqrt[3]{a-bx}}$$

and it is equal to 1. What is the value of the remainder when ab is divided by 5?

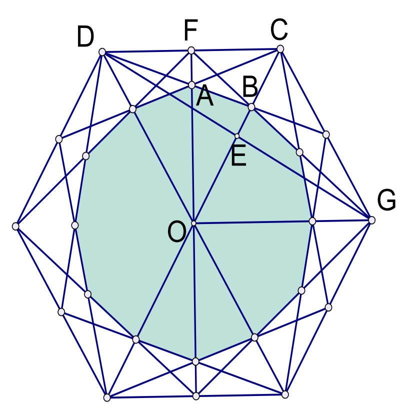
(A) 0 (B) $\boxed{1}$ (C) 2 (D) 3 (E) 4

Solution: In order for the limit to exist, we must have $2\sqrt[3]{a} = 4$ or $a = 8$. This allows us to use L'Hospital's Rule and get

$$L = \lim_{x \to 0} \frac{2x}{-\frac{b}{3}(a+bx)^{-\frac{2}{3}} + \frac{b}{3}(a-bx)^{-\frac{2}{3}}} = \lim_{x \to 0} \frac{2}{\frac{2b^2}{9}(8+bx)^{-\frac{5}{3}} + \frac{2b^2}{9}(8-bx)^{-\frac{5}{3}}} = \frac{16 \cdot 9}{b^2} = 1.$$

Hence, $b = 12$ and so $ab = 96 = 19 \cdot 5 + 1$ which gives the answer B. ■

19. In the adjoining figure, we have a regular hexagon and the midpoints of its sides are connected as shown with some of its vertices. At the intersection of all the line segments is the shaded dodecagon. What part of the hexagon's area has been shaded?



(A) $\frac{5}{11}$ (B) $\boxed{\frac{8}{15}}$ (C) $\frac{3}{7}$

(D) $\frac{4}{7}$ (E) $\frac{1}{2}$

Solution: In the adjoining figure we added some notation that we need. Due to symmetry of this figure, it is clear that the ratio between the area of the dodecagon and the area of the hexagon is the same as $\frac{[OAB]}{[OCF]}$. Using the formula of the area for a triangle, we see that

$$\frac{[OAB]}{[OCF]} = \frac{OB \cdot OA \sin \widehat{AOB}}{OC \cdot OF \sin \widehat{AOB}} = \frac{OB \cdot OA}{OC \cdot OF} = \frac{OB}{OC} \cdot \frac{OA}{OF}.$$

In the triangle DCG the medians intersect at point B which is $1/3$ to E and $2/3$ to C. This gives $\frac{OB}{OC} = \frac{2}{3}$. Using the Ceva's Corollary stated on page II of this test, applied to the triangle ODC and Cevians passing through A, we obtain

$$\frac{OA}{AF} = 2\frac{OB}{BC} = 4.$$

This shows that $\frac{OA}{OF} = \frac{4}{5}$. Hence, we get

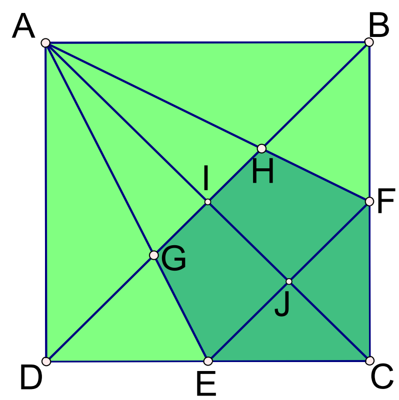
$$\frac{[OAB]}{[OCF]} = \frac{OB}{OC} \cdot \frac{OA}{OF} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}.$$

Therefore, the correct answer is B. ■

20. In the adjoining figure, we have a square $(ABCD)$ and the points E and F are the midpoints of two of its sides. What is the ratio $\frac{[ABCD]}{[CEGHF]}$?

(A) $\boxed{3}$ (B) $\frac{7}{2}$ (C) $\frac{10}{3}$

(D) $\frac{11}{4}$ (E) $\frac{16}{5}$

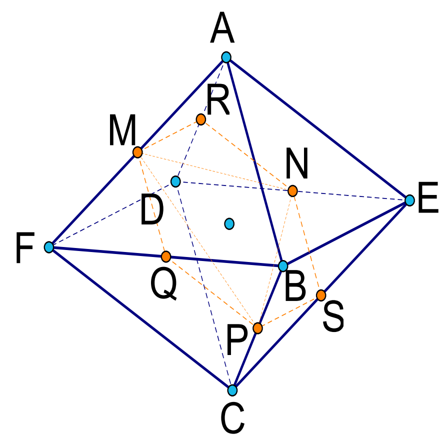


Solution: In the adjoining figure we added some notation that we need. Due to symmetry of this figure, it is clear that the ratio between the area of the square and the area of the shaded pentagon is the same as $\frac{[ABC]}{[IHFC]}$. We see that $[IHFC] = [AFC] - [AHI]$. In the triangle ABC, $\overline{AF}$ and $\overline{BI}$ are medians. Therfore, thay intersect at H which is $1/3$ to I and $2/3$ to B on $\overline{BI}$. Then, we have $[AHI] = \frac{1}{3}[AIB] = \frac{1}{6}[ABC]$. Then we can calculate

$$[IHFC] = [AFC] - [AHI] = \frac{1}{2}[ABC] - \frac{1}{6}[ABC] = \frac{1}{3}[ABC].$$

This implies that the answer is A. ■

21. In the accompanying figure we have a regular octahedron ABCDEF of side-lengths equal to 4. The point M is the midpoint of the side $\overline{AF}$, the point N is the midpoint of the side $\overline{DE}$ and the point P is the midpoint of $\overline{BC}$. The triangle MNP determines a plane which intersects the faces of the octahedron at a polygon $\mathcal{P}$ (not drawn in this figure). What is the area of $\mathcal{P}$?



(A) $2\sqrt{3}$ (B) $4\sqrt{3}$ (C) $8\sqrt{3}$

(D) $\boxed{6\sqrt{3}}$ (E) $10\sqrt{3}$

Solution: In the adjoining figure we added some notation that we need. The polygon $\mathcal{P}$ is now shown in the figure. We can think of the vertices of the octahedron as having coordinates in $\mathbb{R}^3$: $F(2\sqrt{2},0,0)$, $E(-2\sqrt{2},0,0)$, $B(0,2\sqrt{2},0)$, $D(0,-2\sqrt{2},0)$, $A(0,0,2\sqrt{2})$ and $C(0,0,-2\sqrt{2})$. Then M, N and P coordinates can be calculated as usual: $M=(\sqrt{2},0,\sqrt{2})$, $N=(-\sqrt{2},-\sqrt{2},0)$, and $P=(0,\sqrt{2},-\sqrt{2})$. These points are all on the plane of equation $x-y-z=0$. But also are the points obtained by reflection through the origin $S=-M$, $Q=-N$ and $R=-P$. Hence, the polygon $\mathcal{P}$ is $MNPQRS$ and it is easy to see this is an regular hexagon with sides lengths equal to 2. Therefore its area is $6\cdot\frac{4\sqrt{3}}{4}=6\sqrt{3}$. Answer D. ■

22. Suppose that x is an angle satisfying the following trigonometric equation

$$59(\tan x+\cot x)+\sec x+\csc x=120.$$

There are two possible values of $\tan\frac{x}{2}$, say a and b. What is the value of ab?

(A) $\boxed{6}$ (B) 12 (C) -14 (D) -30 (E) 10

Solution: The equality is equivalent to $59(\sin^2 x+\cos^2 x)+\sin x+\cos x=120\sin x\cos x$. Let t be equal to $\sin x+\cos x$. The equality above can then be written as $120\sin x\cos x=59+t$. On the other hand $t^2=1+2\sin x\cos x$. Eliminating $\sin x\cos x$ we get an equation in t:

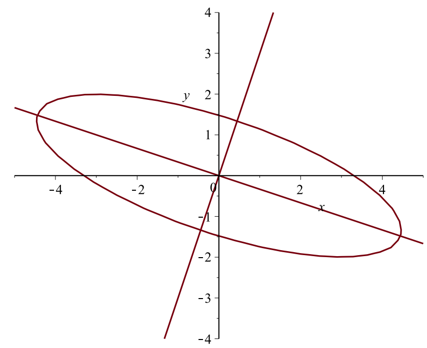
$$60t^2=60+59+t \quad\Leftrightarrow 60t^2-t-119=0\Leftrightarrow(12t-17)(5t+7)=0.$$

This gives two solutions for t. We know that $t^2=1+2\sin x\cos x\le 1+\sin^2 x+\cos^2 x=2$ but $\left(\frac{17}{12}\right)^2=\frac{289}{144}>2$. This leaves only one option for t: $t=\frac{-7}{5}$. Using Weierstrass substitution $u=\tan\frac{x}{2}$ we have $\sin x=\frac{2u}{1+u^2}$ and $\cos x=\frac{1-u^2}{1+u^2}$. We then get the equation in u: $\frac{2u+1-u^2}{1+u^2} = -\frac{7}{5}$ or $10u + 5 - 5u^2 = -7 - 7u^2$. Finally this reduces to $2u^2 + 10u + 12 = 0$ or $u^2 + 5u + 6 = 0$. Hence $u_1 = -2$ and $u_2 = -3$. Therefore the correct answer is A. ■

23. What is the smallest value of $x^2 + y^2$ if $3x^2 + 4xy + 6y^2 = 7$?

    (A) $\boxed{1}$ (B) 2 (C) 3

    (D) 4 (E) 5



Solution: We think of a point $(x, y(x))$ on the curve (see figure above) $3x^2+4xy+6y^2 = 7$, and as a function of x we need to minimize $x^2 + y(x)^2$. Hence we look for critical points, or $2x + 2yy' = 0$. This implies $y' = -x/y$. Using implicit differentiation we get $6x + 4y + 4xy' + 12yy' = 0$ or $2y^2 - 2x^2 - 3xy = 0$. This homogeneous equation can be solved by factorization $(y - 2x)(2y + x) = 0$. Then $y = 2x$ or $y = -x/2$. So, we need to look where the given curve intersects these two lines. First, we look at $y = 2x$: $3x^2 + 4x(2x) + 6(4x^2) = 7$ or $x^2 = \frac{1}{5}$. This gives $x^2 + y^2 = 1$. If we calculate the other intersection points we obtain $x^2 + y^2 = 14$. Therefore, the minimum is 1 and A is the correct answer. ■

24. Find the sum of all fractions $\frac{p}{q}$ between 0 and 1 including 0 and 1, that have denominator 2021 when expressed in lowest terms.

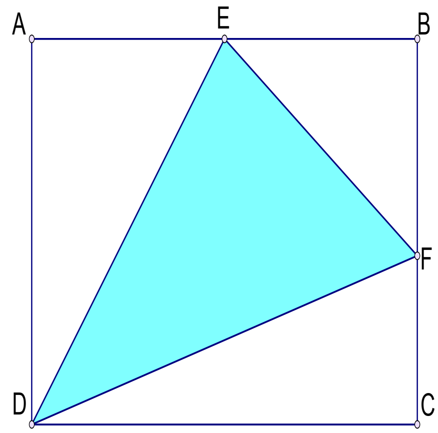
    (A) 968 (B) $\boxed{966}$ (C) 924 (D) 984 (E) 916

Solution: Every fraction in lowest terms which has a denominator of 2021 is of the form $\frac{n}{2021}$ with $n \in \{1, 2, 3, ..., 2021\}$ relatively prime with 2021. Since $2021 = 43 \cdot 47$ is the prime decomposition of 2021, the number of relatively prime (with 2021) numbers $n \in \{1, 2, 3, ..., 2021\}$ is given by Euler's formula $\phi(n) = (43 - 1)(47 - 1) = 1932$. We notice that if n is such a number then $2021 - n$ is also relatively prime with 2021. Hence, the sum of all these numbers is $2021 \cdot 1932 \cdot \frac{1}{2}$. Then the sum of all fractions as required in the problem is $1932 \cdot \frac{1}{2} = 966$. ■

25. In the adjoining figure, we have a rectangle $ABCD$ with sides $AB = 24$ and $BC = 16$. Triangles ADE and CDF have perimeters 48 and 56 respectively. What is the area of DEF?

(A) $\boxed{150}$ (B) 200 (C) 250

(D) 156 (E) 253



Solution: If we denote by x the length of the segment AE we have the equation $x + 16 + \sqrt{x^2 + 16^2} = 48$. This can be solved easily and one obtains $x = 12$. Hence $EB = 12$. Similarly, setting $FC = y$ we have the equation $y + 24 + \sqrt{y^2 + 24^2} = 56$. This implies $y = 7$ and so $BF = 9$. We have $ED = 20$, $DF = 25$ and $EF = 15$ using the Pythagorean theorem. To get the area we can use Heron's formula or we can subtract the areas of the triangles ADE, EBF and CDF form the area of the rectangle. Either way we get $[DEF] = 150$. So, the correct answer is A. ■

26. The following is an identity in a, b and c, i.e., for some fixed real values of $\mathbf{x}$ and $\mathbf{y}$ that have to be determined, it is true for all real values of the variables a, b and c:

$$(a+b+c)^5 = a^5 + b^5 + c^5 +$$

$$\mathbf{x}(a+b)(b+c)(a+c)[a^2 + b^2 + c^2 + \mathbf{y}(ab + bc + ca)]$$

What is $\mathbf{x} - \mathbf{y}$?

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: If we let $a \to -b$ in

$$\frac{(a+b+c)^5 - (a^5 + b^5 + c^5)}{a+b}$$

using L'Hopital's Rule, we obtain $5(c^4 - b^4)$ and from the identity, this limit is equal to $\mathbf{x}(b+c)(c-b)[2b^2 + c^2 - \mathbf{y}b^2]$. We conclude that

$$(c^2 - b^2)[(5 - \mathbf{x})c^2 + (5 - \mathbf{x}(2 - \mathbf{y})b^2] = 0$$

for every b and c. Therefore, $\mathbf{x} = 5$, $\mathbf{y} = 1$ and the answer is D. ■

27. The continuous function f defined on $\mathbb{R}$ satisfies

$$\int_0^x (x-t+1)f(t)dt = x^2 + 2x + 2\sin x + e^x - 1 \quad \text{for all} \quad x \in \mathbb{R}. \tag{2}$$

Find $f(0)$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

Solution: Let us denote by $G(x) = \int_0^x f(t)dt$. The equation (2) can be written as

$$G(x) - \frac{1}{x}\int_0^x (t-1)f(t)dt = x + 2 + 2 \cdot \frac{\sin x}{x} + \frac{e^x - 1}{x}.$$

Letting $x \to 0$, using L'Hospital's Rule and the Fundamental Theorem of Calculus, we obtain $f(0) = 2 + 2 + 1 = 5$ which gives the answer E. ■

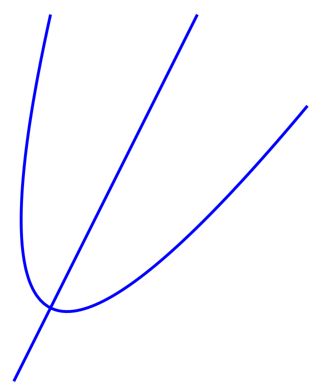
28. The equation

$$(y - 3x)^2 = y - x$$

is a parabola with the vertex at $V(\alpha, \beta)$. What is $\frac{\alpha}{\beta}$?

(A) 1 (B) $\boxed{2}$ (C) 3

(D) 4 (E) 5



Solution: The equation of the axis of symmetry of the parabola, say $y = mx + b$, cannot have $m \neq 3$, otherwise the number of intersection points with the parabola leads to the quadratic equation,

$$[(m-3)x + b]^2 + = mx + b - x,$$

which means two or zero intersections (the case of one solution with multiplicity two means the line is tangent to the parabola and this excluded for the axis of symmetry), while the axis of symmetry has only one intersection point with the parabola (the vertex). So, we must have $m = 3$. Hence, we only need to determine b. This means in particular that the slope of the tangent line at the vertex is $-\frac{1}{3}$. If we differentiate implicitly, the equation of the parabola, we obtain at the vertex

$$2(y - 3x)(y' - 3) = y' - 1 \;\Leftrightarrow\; 2b(-\frac{1}{3} - 3) = -\frac{1}{3} - 1 \Rightarrow b = \frac{1}{5}.$$

Hence, at the vertex, $\frac{1}{25} = 3x + b - x$ or $x = \alpha = \frac{1}{2}(\frac{1}{25} - \frac{1}{5}) = -\frac{2}{25}$ and then $\beta = 3\alpha + b = -\frac{1}{25}$. Therefore, $\frac{\alpha}{\beta} = \boxed{2}$. ■

29. How many different ways can 2021 be written as a sum of at least two positive consecutive integers?

(A) 4 (B) $\boxed{3}$ (C) 2 (D) 1 (E) 0

Solution: Let us say we have the equation $2021 = a + (a+1) + ... + (a+k-1)$ where $k > 1$ is the number of consecutive terms and $a \in \mathbb{N}$. Then this is equivalent to $2021 = ka + \frac{(k-1)k}{2}$ or

$$43 \cdot 47 = k(a + \frac{k-1}{2}).$$

We have two cases. First, let us assume that k is an odd number. Then $a + \frac{k-1}{2}$ is an integer also, and so k can be equal to 43, 47 or 2021. We obtain only two representations:

$$26 + 27 + ... + 68 = 2021 \quad \text{and}$$

$$20 + 21 + ... + 66 = 2021.$$

In the case k even, i.e., $k = 2m$ for some $m \in \mathbb{N}$ we have the equation $2021 = m(2a + 2m - 1)$ and so m can be equal to 1, 43, 47 or 2021. We only get a new solution if $m = 1$:

$$1010 + 1011 = 2021.$$

Therefore, the correct answer is B. ■

30. Computing the limit

$$L := \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{4n^2 - k^2}$$

we obtain $L = \frac{\sqrt{3}}{m} + \frac{\pi}{n}$ for some positive integers m and n. What is the remainder of 2021 divided by mn ?

(A) 1 (B) 3 (C) $\boxed{5}$ (D) 7 (E) 9

Solution: We use the Riemann Sums definition of the definite integral and see that the limit is equal to

$$L = 4 \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n} \sqrt{1 - \frac{k^2}{4n^2}} = 4 \int_0^{\frac{1}{2}} \sqrt{1 - x^2} dx.$$

To compute the last integral we substitute $x = \sin t$:

$$4 \int_0^{\frac{1}{2}} \sqrt{1 - x^2} dx = 4 \int_0^{\frac{\pi}{6}} (\cos t)^2 dt = 2 \int_0^{\frac{\pi}{6}} (1 + \cos 2t) dt = \frac{\pi}{3} + \sin\left(2 \cdot \frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

Hence, $m = 2$ and $n = 3$. Therefore $2021 = 336 \cdot 6 + 5$ and so C is the answer here. ■