Forty-Fifth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

The problems are written in increasing order of difficulty. So, for tiebreaker problems, we begin with the last problem and go in order toward the first problem.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

All numbers when referred to their digits are considered written in base 10 unless otherwise stated. Important facts and notations to consider:

- $2019 = 3(673)$ (prime factorization)
- $673 = 12^2 + 23^2 = 5^2 + 2(18^2) = 2(19^2) - 7^2$
- $2019^2 = 1155^2 + 1656^2$ (unique writing up to order and change of signs)
- $[XYZW...]$ means the area of the polygon $XYZW...$

1. How many solutions (x, y), does the system of equations

$$\begin{cases} |x| + |y| = 2, \\ x^2 - y^2 = 1, \end{cases}$$

have?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. What is the largest possible remainder that can be obtained when a two-digit number is divided by the sum of its digits?

(A) 10 (B) 11 (C) 12 (D) 15 (E) 17

3. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin being used?

(A) 35 (B) 37 (C) 39 (D) 41 (E) 43

4. The number $\sqrt{24 + \sqrt{572}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, for two positive integers a and b. What is $|a - b|$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. How many prime numbers divide $5^6 - 1$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

6. Find the remainder of the division $(x^4 - 4x^3 + 6x^2 - 3x) \div (x^2 - 2x + 1)$.

(A) x (B) $x - 1$ (C) $x - 2$ (D) $x - 3$ (E) $x - 4$

7. A fair coin is flipped three times. What is the probability that at least one head will be thrown?

(A) 0.125 (B) 0.25 (C) 0.75 (D) 0.9 (E) 0.875

8. When expanded as a decimal, the fraction $\frac{1}{97}$ has a repetend (the repeating part of the decimal) that begins right after the decimal point and is 96 digits long:

$$\frac{1}{97} = 0.\underbrace{01030927835...x67}_{96\ digits}...$$

If the last three digits of the repetend are $x67$, what is the digit x?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. What is the number of solutions in x (real values) of the equation

$$|x - 3| + |3x - 5| + |2x - 7| = 4?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

10. Assuming that t is the only real solution of the cubic equation $x^3 = x + 2$, then there are unique integers m, n and p so that $t^8 = mt^2 + nt + p$. Find $m + n - p$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

11. The sum

$$2 \cdot \left[\cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ\right]$$

equals a two-digit number ab. What is $b - a$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

12. The recurrent sequence $\{x_n\}$ satisfies the recurrence $x_{n+1} = 2x_n(1-x_n)$ for every $n \geq 1$ and $x_1 = 1/2019$. Knowing that $\{x_n\}$ is convergent to L, what is L^{-1}?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

13. The following quadratic equations $15x^2 - 19x + 6 = 0$ and $21x^2 - 17x + 2 = 0$ have a common solution. What is the sum of the other two solutions?

(A) $\frac{24}{35}$ (B) $\frac{22}{35}$ (C) $\frac{28}{35}$ (D) $\frac{20}{35}$ (E) $\frac{26}{35}$

14. The numbers a_1, a_2, ..., a_6 are positive consecutive integers whose sum is 2019. What is the value of

$$\frac{6a_4 - a_1 - 2a_2 - 3a_3}{10}?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

15. The equation in x,

$$(x-1)^5 + (x-2)^5 + (x-3)^5 + (x-4)^5 + (x-5)^5 = 0$$

has five solutions (including the pure complex ones). What is their product?

(A) 875 (B) 885 (C) 895 (D) 785 (E) 685

16. If $\log_A B$ means the logarithm of B in base A, solve the equation $\log_9 x + \log_{x^2} 3 = 1$ for x.

(A) 9 (B) 27 (C) 81 (D) 3 (E) 1

17. Find the sum of the number $N = 1234$ and all other four digit numbers obtained by permuting the digits of N in all possible ways.

(A) 33330 (B) 66660 (C) 999990 (D) 55550 (E) 11110

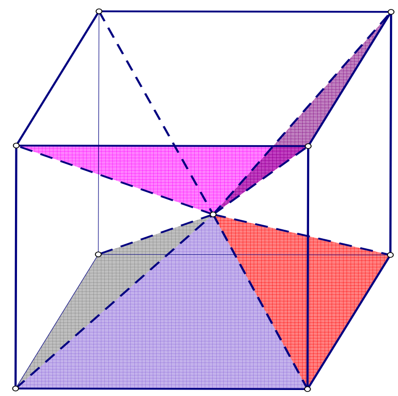
18. The cubic equation $x^3 - 12x^2 + 44x + a = 0$ has three solutions in arithmetic progression. Find a.

(A) 50 (B) -48 (C) -55 (D) 20 (E) -35

19. A cube is dissected into six pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 2, 5, 10, 11 and 14. What is the volume of the sixth pyramid?

(A) 4 (B) 5 (C) 6

(D) 7 (E) 8



20. The positive integer N has exactly six distinct (positive) divisors including 1 and N. The product of only five of these divisors is 648. What is the value of the sixth missing divisor?

(A) 18 (B) 9 (C) 6 (D) 2 (E) 1

21. Suppose that a and b are positive integers such that $2019 > a > b \geq 1$, and the expression

$$\frac{a+b}{a-b} + \frac{a-b}{a+b}$$

attains its smallest possible value. What is the units digit of $a + 3b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

22. A natural number is called a *perfect power* if it is equal to a^b for some a and b natural numbers $b \geq 2$. The list of perfect powers begins with 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, and so on. The sum S_n of the first n perfect powers gives a sequence $S_1 = 1$, $S_2 = 1 + 4 = 5$, $S_3 = 1 + 4 + 8 = 13$, etc. Knowing that $S_n = 2019$ for some n, find the sum of the digits of n.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

23. A box has 5 red balls and 3 blue balls. Bill picks one ball at the time, from the box, without replacement, until all of balls of the same color are out. If $\frac{p}{q}$ is the probability that the last ball Bill picks is blue, written as a reduced fraction, what is $q - p$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. The real valued function f defined by

$$f(x) = \sum_{j=0}^{2019} |x - j|, x \in \mathbb{R},$$

has a minimum value of M^2. What is the sum of the digits of M?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

25. For every real number x, a vertical segment connecting the graphs of

$$y = |x| \text{ and } y = -x^2 - 3x - 2$$

can be drawn (segment $\overline{AB}$ with $A = (x, |x|)$ and $B = (x, -x^2 - 3x - 2)$). Find the smallest possible length of these vertical segments $\overline{AB}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

26. The Golden ratio ϕ is the famous number which equals $\frac{1+\sqrt{5}}{2}$. Consider the real valued function f defined for all $z \neq \phi$, by $f(z) = \frac{1}{\phi - z}$. The composition of f with itself, n-times is denoted by $f^{[n]}$. For example, $f^{[2]} = f \circ f$, $f^{[3]} = f \circ f \circ f$, etc. Calculate $f^{[2019]}(2)$.

(A) $3 - 2\phi$ (B) 2 (C) $\frac{2\phi - 1}{2}$ (D) $\frac{\phi + 2}{5}$ (E) $\phi + 1$

27. Assume that a is a positive real number. A recurrent sequence $\{x_n\}$ is defined by $x_1 = 1$ and

$$x_{n+1} = \frac{x_n}{1 + ax_n}$$

for all natural numbers $n \geq 1$. Knowing that

$$x_{674} = \frac{1}{2020}$$

find a.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

28. We have a function f defined for all real numbers x by the functional equation

$$f(x) + xf(-x) = 1.$$

Knowing that the positive integer M, which is equal to

$$2019 \cdot f\left(\frac{12}{23}\right),$$

is a 3-digit number (in base 10), what is its first digit (the hundreds place)?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

29. Find the number of positive integer solutions (x, y), of the following Diophantine equation:

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

30. The expression in terms of the real valued variable x,

$$\sec^2(x) + 2\csc^2(x) + 3\sec^2(x)\csc^2(x)$$

has minimum value equal to $m+4\sqrt{n}$ over the interval $(0, \frac{\pi}{2})$, for some natural numbers m and n. What is $m - n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

31. Assume that x is not equal to zero. If we eliminate α from the two equalities

$$\begin{cases} x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \\ x\cos 2\alpha + y\sin 2\alpha = 0, \end{cases}$$

we get $4x^2 + 4y = n$. Find n.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

32. A number t is randomly chosen (uniform distribution) from the unit interval $[0,1]$. Calculate the expectation of the random variable $X(t) = \max\{t, 1-t\}$.

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{4}{5}$ (D) $\frac{3}{5}$ (E) $\frac{3}{4}$

33. The equation $9x^3 = a + \ln x$ has a unique positive solution in x given this unique particular positive real number a. What is the value of $e^{a-\frac{1}{3}}$?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

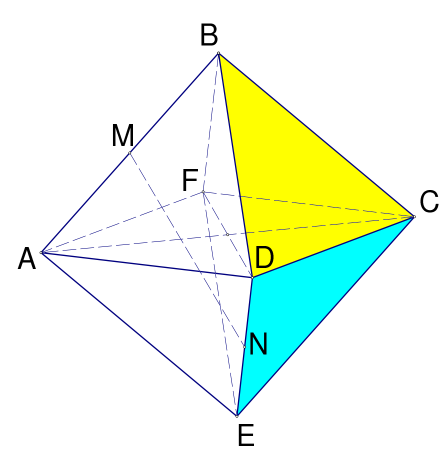
34. The graph of the function g is the reflection of the graph of $f(x) = 2x^3 - 3x^2 + 3x$ defined for all real x, across the line $y = x$. What is $g'(1)$ (i.e. $\frac{dg}{dx}(1)$) ?

(A) $\frac{3}{2}$ (B) $\frac{1}{3}$ (C) $\frac{3}{4}$ (D) $\frac{2}{5}$ (E) $\frac{2}{3}$

35. In the accompanying figure we have a regular octahedron ABCDEF of side-lengths equal to 4. The point M is the midpoint of the side $\overline{AB}$ and the point N is the midpoint of the side $\overline{DE}$. How long is the segment $\overline{MN}$ (same units)?

(A) $2\sqrt{5}$ (B) 4 (C) $2\sqrt{3}$

(D) $2\sqrt{6}$ (E) $2\sqrt{2}$



36. Find the limit of

$$\lim_{x\to 0}\frac{\sqrt[3]{5x+8}-2}{x}.$$

(A) 5/12 (B) 7/12 (C) 1/12 (D) 5/4 (E) 5/3

37. Knowing that every solution of the equation $(\cos x)^2-(\cos x)-1=0$ is a solution of the equation $a(\cos 2x)^2+b(\cos 2x)-1=0$, what is $a+b$?

(A) -1 (B) -2 (C) -3 (D) -4 (E) -5

38. How many integer ordered pairs (x, y) of solutions does the Diophantine equation $x^2+y^2=x^3+y^3$ have ?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

39. For positive real numbers a and b, the equation $x+\ln(ab)=\ln(x+a)+\ln(x+b)$ has a unique real solution in x. What is the value of $1/a+1/b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

40. Two points are chosen at random on a circle of radius 1. What is the expectation of the distance between them ?

(A) 2/3 (B) $\frac{\pi}{3}$ (C) $\frac{3}{\pi}$ (D) $\frac{4}{\pi}$ (E) $\frac{3}{2}$

41. It is known that every positive integer can be written as a sum of non-consecutive Fibonacci numbers ($F_1=1$, $F_2=1$, $F_{n+1}=F_n+F_{n-1}$, $n\geq 2$) in a unique way. Taking into consideration this writing for 2019,

$$2019=F_{n_1}+F_{n_2}+\cdots+F_{n_k},$$

find k.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

42. How many positive integers n have the property that both $\sqrt{2n-4030}$ and $\sqrt{2n+4030}$ are integers?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

43. Find the smallest natural number n such that there exists a choice of signs that makes the following equality true

$$\pm 1 \pm 2 \pm 3 + \cdots \pm n = 2019.$$

(A) 61 (B) 62 (C) 63 (D) 64 (E) 65

44. Find the closest integer to the sum

$$\sum_{k=1}^{2019} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}.$$

(A) 2019 (B) 2020 (C) 2021 (D) 2022 (E) 2023

45. The value of the integral

$$\int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1+\sqrt{2x}} \sin(\pi x) dx$$

is equal to $\frac{\sqrt{n}}{m\pi}$ for some $m, n \in \mathbb{N}$. Find $3n - m$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

46. The positive integers x, y, z satisfy $x < y < z$ and

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}.$$

Find the last digit of $x - y + z$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

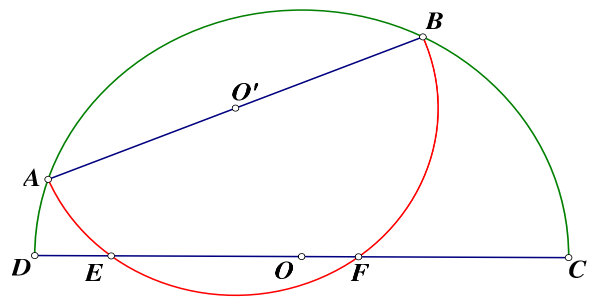
47. The Taylor expansion of $f(x) = \frac{(x+1)(3-2x)}{(1-x)^3}$ around $x = 0$ has as one of its terms equal to $2019x^k$ for some positive integer k. The number k is a two digit number if written in base 10, i.e., $k = 10a + b$ with $a, b \in \{0, 1, ..., 9\}$. What is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

48. In the adjacent figure (not drawn to scale), we have two semi-circles with centers O and O'. Knowing that $DE = 3$, $EF = 17$ and $FC = 27$, what is the sum of the digits of AB?

(A) 1 (B) 2 (C) 3

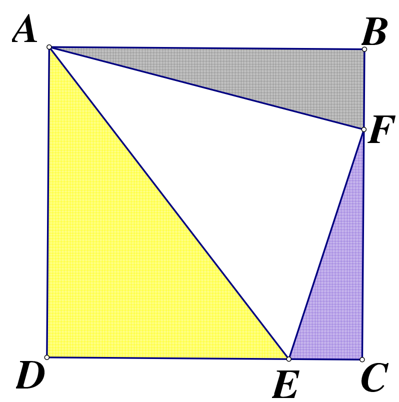
(D) 4 (E) 5



49. In the adjacent figure $ABCD$ is a square and points E and F are on its sides $\overline{DC}$ and $\overline{BC}$ respectively. By $[XYZ]$ we denote the area of the triangle XYZ. Knowing that $[ADE] = 21$, $[ECF] = 2$ and $[ABF] = 18$, find $[AEF]$.

(A) 11 (B) 12 (C) 13

(D) 14 (E) 15



50. A rectangular piece of paper $ABCD$ with side lengths $AB = 34$ cm and $AD = 32$ is folded along the dashed line segments $\overline{FB}$ and $\overline{EC}$, such that the points A and D arrive at the same point P. Find the area of the trapezoid $EFBC$.

(A) 799 (B) 800 (C) 801

(D) 802 (E) 803