NINETEENTH ANNUAL COLUMBUS INVITATIONAL MATHEMATICS TOURNAMENT

Sponsored by

THE COLUMBUS COLLEGE

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

March 6, 1993

* * * * * * * * * * * * * * * *

The Columbus College Mathematics and Computer Science faculty welcome you to this year's tournament and to our campus. We hope that you will be successful on this test.

INSTRUCTIONS: This is a 90-minute, 50-problem, multiple-choice examination. There are five (5) possible responses to each question. You are to select the one (1) "best" answer to each. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the choice you have made on the test booklet. After you have worked all of the problems that you can work, carefully transfer your answers to the score sheet. Darken completely the blank corresponding to the letter of your response to each question. Mark your answers boldly with a No. 2 pencil. If you must change an answer, completely erase your first choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, -3 for each incorrect selection and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used for tie-breakers for individual awards. These problems, in the order in which they will be examined, are: 13, 17, 23, 36, and 40.

All pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. Your student identification number and your school number must be encoded correctly on your score sheet.

When you complete your test, bring your pencil, scratch paper and answer sheet to the Test Monitor. You may leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. **Please do not congregate outside the doors to the testing area.** You may keep your copy of the test. Your sponsor will have a copy of the answers.

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO DO SO

1. The difference of a positive number and its reciprocal is 21/10. Find the sum of the number and its reciprocal.

a) 24/10 b) 31/10 c) 9/5 d) 29/10 e) 13/5

2. Represent the solution to the following inequality in interval notation:

$$-1 \le \frac{1 - 2x}{3} \le 4$$

a) [-5/2,2] b) [-2,9/2] c) [-11/2,2]
d) [-3/2,4/3] e) [-3,9/2]

3. A car enters a freeway going 75 kilometers per hour (km/hr) and fifteen minutes later a second car enters at the same location going 90 km/hr. How many minutes after the second car enters the freeway will it overtake the first car?

a) 30 b) 75 c) 45 d) 90 e) 60

4. What is the frequency (in oscillations per second) if the simple harmonic motion of an object is represented by $d = 2 \cos(4t)$, where d is the distance (in meters) of an object from its rest position after t seconds?

a) $\pi/2$ b) 4 c) 1/4 d) $2/\pi$ e) 8π

5. If $\sin(2A) = -1$, $\pi/2 \le A \le \pi$, and $\cos B = \sin(2B)$, $\pi/2 < B < \pi$, find A + B.

a) $19\pi/12$ b) $5\pi/6$ c) $17\pi/12$ d) $3\pi/2$ e) $3\pi/4$

6. Decompose $\dfrac{7 - 4x}{6x^2 + 13x - 5}$ into the form $\dfrac{A}{3x - 1} + \dfrac{B}{2x + 5}$.

Find A + B.

a) -5 b) -1 c) 3 d) 6 e) -4

7. How many gallons of water should be added to 1 gallon of pure antifreeze to obtain a solution that is 40% antifreeze?

a) 2   b) 2.5   c) 1.5   d) 1.75   e) 2.25

8. Find the range of the quadratic function $y = -5x^2 + 20x - 13$.

a) $(-\infty,7]$ b) $[2,\infty)$ c) $(-\infty,-13]$ d) $(-\infty,\infty)$ e) $[-7,\infty)$

9. Solve for B: $A = \frac{H(b + B)}{2}$.

a) $B = \frac{2A}{bH}$ b) $B = \frac{2A - bH}{H}$ c) $B = 2A - b$

d) $B = 2A - Hb$ e) $B = 2b - A/H$

10. Which of the following functions has amplitude 3 and period 6?

a) $y = 3 \sin (3x - 2)$ b) $y = 3 \tan ((\pi/3)x)$
c) $y = 3 \sin (12\pi x + \pi)$ d) $y = 3 \cos ((\pi/3)x - 2)$
e) $y = \frac{1}{3} \cos ((2\pi/3)x - \pi)$

11. If $(-\sqrt{3},-1)$ are the rectangular coordinates of a point, find a pair of polar coordinates for this point.

a) $(-2,\pi/3)$ b) $(2,3\pi/4)$ c) $(2,4\pi/3)$

d) $(-2,\pi/6)$ e) none of the preceding

12. Which of the following is a vertical asymptote for

$$y = \frac{x^2 - 21}{12x^2 - 25x - 75}?$$

a) $x = 5/3$ b) $y = -5/3$ c) $x = 0$ d) $x = 15/4$ e) $x = \sqrt{21}$

13. Solve for x: $\log_{\sqrt{2}} x = -6$.

a) 1/8 b) $\sqrt{3}$ c) 12 d) -8 e) 1/32

14. A life insurance table indicates that a man who is now x years old can expect to live y years longer. Suppose x and y are linearly related and that y = 50 when x = 24 and y = 20 when x = 60. At what age may a man expect to live 30 years longer?

a) 42 b) 53 c) 37 d) 48 e) 45

15. A ship leaves port and travels 55 km due south. It then changes course and sails 41 km bearing N 27°15′ W. How many kilometers is the ship from port at this point? (Round your answer to the nearest hundredth.)

a) 25.18 b) 21.12 c) 36.66 d) 96.00 e) 26.39

16. A goat is tethered to the corner of a square building that is 20 feet on a side by a rope that is 30 feet long. Calculate its grazing area in square feet.

a) 675π b) 725π c) 900π d) 600π e) 450π

17. Find the equation of a circle having (3,-5) and (-3,-1) as endpoints of a diameter.

a) $x^2 + y^2 = 6x - 4y - 4$ b) $x^2 + y^2 = 2y + 44$

c) $x^2 + y^2 = 6x - 4y$ d) $x^2 + y^2 = -6y + 43$

e) $x^2 + y^2 = -6y + 4$

18. Simplify the following expression: $\dfrac{(2a^b)^2\ (-4a^{2+b})}{8a^{4b}}$.

a) $-2a^{b+2}$ b) $2a^{2-b}$ c) $-2a^{b-2}$ d) $\dfrac{-a^{b+2}}{2}$ e) $-2a^{2-b}$

19. If the surface area of a certain sphere is the same as the area of a circle of radius 6 centimeters, find the radius (in centimeters) of the sphere.

a) 12 b) 6 c) 3 d) 2 e) 9

20. Determine the area (in square units) of the region which is the intersection of the regions $x^2 + y^2 \le 16$ and $y \ge |x|$.

a) π b) 4π c) 2π d) 8π e) 16π

21. Find the sum of the following infinite series:

$$1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \ldots$$

a) 1 b) $\frac{511}{512}$ c) 2 d) $\frac{15}{16}$ e) $\frac{1023}{1024}$

22. A bicycle wheel of diameter 20 inches is rolled through an angle of 720°. How far (to the nearest inch) has the wheel moved?

a) 20π b) 126 c) 250 d) 82 e) 40

23. Let n(A) represent the number of elements in a finite set A. If A, B and C are three finite subsets of a universal set U, and $n(A) = n(B) = n(C) = 16$, $n(A \cap B) = 11$, $n(A \cap C) = 5$, $n(B \cap C) = 8$, and $n(A \cap B \cap C) = 3$, find $n(A \cup B \cup C)$.

a) 27 b) 30 c) 24 d) 21 e) 15

24. If $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$, find

$(\sinh x)^2 - (\cosh x)^2$.

a) .5 b) $2e^2$ c) -1 d) 0 e) 1

25. If $8^x = 27$, find 4^x.

a) $\log_2 3$ b) 3^2 c) $\log_8 27$ d) 3 e) $\log_4 27$

26. It took exactly the same amount of paint to cover the top and bottom of a right circular cylindrical tank as it did to cover the sides. If the tank is h feet high, express the radius r of the tank in terms of h.

a) h b) $2h$ c) h^2 d) $2\pi h$ e) πh^2

27. The diameter AB of the circle given in Figure (1) has length 12, and CD, which is perpendicular to the diameter, has length 5. Find the length of AD.

a) 4 b) 2.8 c) $6 - \sqrt{11}$ d) 9 e) 3.5

28. To harvest a rectangular field of wheat, which is 720 meters by 960 meters, a farmer cuts swaths around the outside, thus forming a steadily growing border of cut wheat and a steadily shrinking rectangle of uncut wheat in the middle. How wide was the border when the farmer was half through?

a) 120 m b) 480 m c) 174.3 m d) 420 m e) 720 m

29. Find c so that the two solutions of $6x^2 - 13x + c = 0$ are reciprocals.

a) 2/3 b) 1/3 c) 2 d) 6 e) 1/6

30. Figure (2) shows a ring-shaped sidewalk surrounding an irregular pond. We wish to determine the area A of the sidewalk but do not know the radii of the two circles. In fact, we do not even know where their common center is and we have no intention of getting our feet wet. However, we can easily make the measurements a = 5 and b = 7 as shown in Figure (2). Find the area of the sidewalk.

a) 2π b) 3π c) 4π d) 5π e) 6π

31. If $f(x) = x^3 + 1$ and $g(x) = x + 2$, find $g(f(2x))$.

a) x^3+3x+2 b) $8x^3+3$ c) $8x^3+12x^2+6x+3$

d) $8x^3+12x^2+12x+9$ e) x^3+2x+3

32. If $a + b = 1$ and $a^2 + b^2 = 2$, find $a^3 + b^3$.

a) 1 b) 2 c) 2.5 d) 3 e) 1.5

33. If $(.67)^H = .5$, find the value of $32(.67)^{4H}$.

a) 128 b) 64 c) 16 d) 8 e) 2

34. Manhattan Island is said to have been bought from the Indians by Peter Minuit in 1626 for $24. If, instead of making this purchase, Minuit had put the money in a savings account drawing interest at 6 percent compounded annually, what would that account be approximately worth (in billions of dollars) in the year 2000?

a) 3 b) 49 c) 82 d) 61 e) 70

35. Solve for y in terms of x: $\log_a(x + y) = \log_a x + \log_a y$

a) x b) x - 1 c) x/(x - 1) d) 1 - x e) 0

36. Let log represent $\log_{10}$. Evaluate

$$\log(1/2) + \log(2/3) + \log(3/4) + \ldots + \log(98/99) + \log(99/100)$$

a) $-\infty$ b) 10^{-2} c) -2 d) -10 e) -1

37. The collected works of this brilliant Swiss mathematician will fill 74 volumes when completed. No other person has written so profusely on mathematical topics. Remarkably, 400 of his papers were written after he was totally blind. One of his contributions was the introduction of the number e = 2.71828... as the base for natural logarithms. What is his name?

a) Napier b) Euler c) Einstein

d) Cardano e) Euclid

38. The equation $x^4 - 68x^2 + 256 = 0$ has four solutions. The sum of the absolute values of the four solutions is

a) 68 b) 0 c) 256 d) 20 e) 120

39. Let ABCD be the vertices of a rectangle, labeled in cyclic order. Suppose that P is a point in the interior of rectangle ABCD and at distances a, b, c and d from A, B, C and D, respectively. Which of the following must be true?

a) $a^2 + b^2 + c^2 = d^2$ b) $a^2 + b^2 = c^2 + d^2$

c) $(a + c)^2 = (b + d)^2$ d) $a + b + c + d = 1$

e) $a^2 + c^2 = b^2 + d^2$

40. Two ordinary dice are tossed. What is the probability of the number on one of the dice being twice that on the other?

a) 1 b) 5/6 c) 2/3 d) 1/6 e) 7/36

41. Find the area of an isosceles triangle with two sides equal to s when the angle between the two equal sides is denoted by θ.

a) $\frac{s^2}{2}$ b) $s \sin \theta$ c) $\frac{s^2}{2} \sin \theta$ d) s^2 e) $\frac{s^2}{4} \sin \theta$

42. Early in the morning a tree casts a shadow of 20 feet. Later in the day, when the inclination of the sun is twice as large, the shadow is 8 feet. How tall is the tree?

a) 8 ft b) 10 ft c) $5\sqrt{2}$ ft d) 12 ft e) $4\sqrt{5}$ ft

43. If $x_1 = 8500$, $x_2 = 8330$, $x_3 = 8480$, $x_4 = 7960$, and $x_5 = 8030$ constitute a random sample of size 5, and $\overline{x} = (x_1+x_2+x_3+x_4+x_5)/5$, the sample variance, denoted by s^2, is defined by $s^2 = \sum_{i=1}^{5} (x_i - \overline{x})^2/4$. Find s, the sample standard deviation, (to the nearest hundredth).

a) 225.30 b) 8260.45 c) 251.89 d) 63450.92 e) 50760

44. A certain radioactive element has a half-life of 1690 years. Starting with 30 milligrams there will be q(t) milligrams left after t years, where

$$q(t) = 30(.5)^{kt}.$$

How many milligrams (to the nearest hundredth) will be left after 2500 years?

a) 10.76 b) 14.93 c) 0 d) 7.5 e) 18.78

45. A man bought x 39-cent pens and y 69-cent pens for $11.37. Find x + y.

a) 23 b) 27 c) 19 d) 33 e) 17

46. The number of ways in which it is possible to distribute m distinct objects into n identical containers ($m \geq n$), with no container left empty, is

$$\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

This will be denoted by S(m,n) and is called a Stirling number of the second kind. Find S(7,3).

a) 90 b) 65 c) 63 d) 301 e) 25

47. Consider the following Pascal program segment, where i, j, and k are integer variables.

```
For i := 1 to 20 do
   For j := 1 to i do
      For k := 1 to j do
         Write   (i*j + k);
```

How many times is the Write statement executed in this program segment?

a) 210 b) 420 c) 1540 d) 1140 e) 20!

48. Let A = {1,2,3,...,15}. How many subsets of A contain all of the odd integers in A?

a) 2^{15} b) 2^7 c) 2^6 d) $\frac{15!}{6!7!}$ e) $\frac{7!}{3!4!}$

49. Given that the expansion of $1/(1 - x)^3$ is $\sum_{i=0}^{\infty} \binom{2+i}{i} x^i$, find the coefficient of x^{15} in the expansion of

$$(x^3 - 5x)/(1 - x)^3.$$

a) 136 b) -509 c) 91 d) 211 e) -29

50. In an organic laboratory, Kelsey synthesizes a crystalline structure that is made up of 1,000 triangular layers of atoms. The first layer of the structure has one atom, the second layer has three atoms, the third layer has six atoms, and, in general, the nth layer has 1 + 2 + 3 + ... + n atoms (see Figure (3)). How many atoms are there in one of these crystalline structures?

a) 5.005×10^5 b) 5.01501×10^5 c) 1.66167×10^8

d) $4.141712475 \times 10^{10}$ e) 1.67167×10^8

Each problem that I solved became a rule
which served afterwards to solve
other problems.
- Descartes

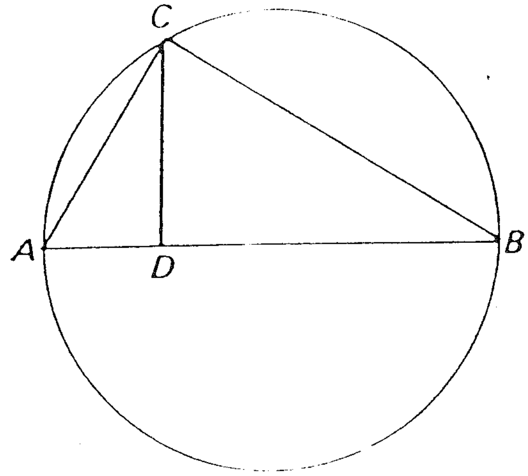

Figure 1

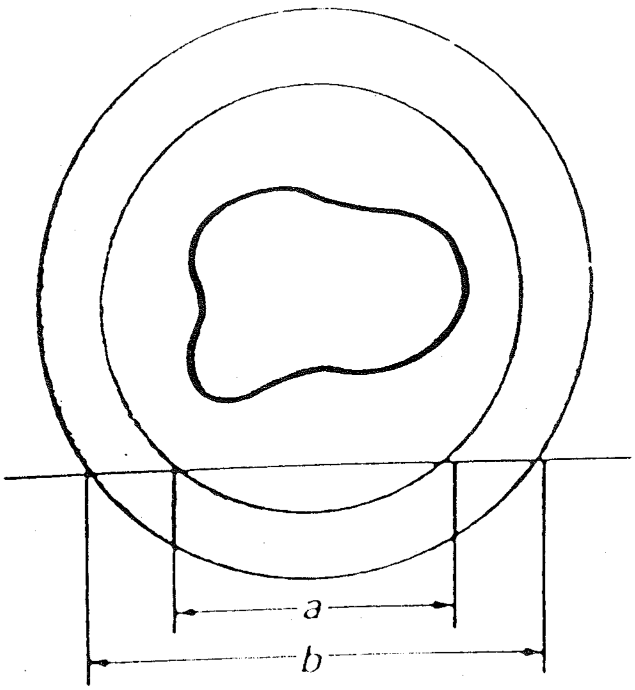

Figure 2

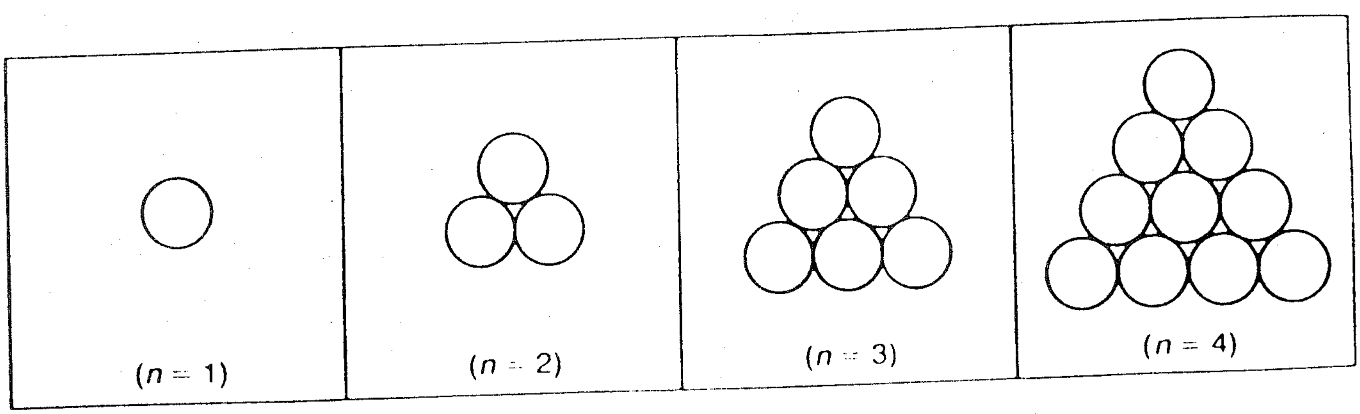

Figure 3