Thirty-fifth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
February 28, 2009

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The Mathematics Department at Columbus State University welcomes you to our campus and to this year's tournament. We wish you success on this contest and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200. Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 13, 15, 16, 17, 23, 27, 28, 29, 34, 36, 38, 39, 41, 45, and 47.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.** When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) Jack had an average score of 85 on his first 4 quizzes. He has an average score of 82 on his first 5 quizzes. What did he receive on his fifth quiz?

A) 60 B) 65 C) 66 D) 70 E) 75

2) For $x \neq 0, \pm 1$, the expression $\dfrac{\dfrac{1}{x^{2007}} - \dfrac{1}{x^{2009}}}{\dfrac{1}{x^{2008}} - \dfrac{1}{x^{2010}}}$ is equivalent to which of the following?

A) x B) $x-1$ C) x^2-1 D) $\dfrac{1}{x}$ E) 1

3) Find the product of the solutions of the equation $\dfrac{3}{x} = \dfrac{8}{x-3} - 1$.

A) -6 B) -9 C) 9 D) $-\dfrac{9}{5}$ E) 10

4) If the surface area of cube A is 64% of the surface area of cube B, then the volume of cube A is what percentage of the volume of cube B?

A) 0.64 B) 0.512 C) 51.2 D) 32 E) 64

5) If the two lines $3y + x + 2 = 0$ and $2y + ax + 3 = 0$ are perpendicular, what is the value of a?

A) $\dfrac{1}{3}$ B) $-\dfrac{1}{3}$ C) 6 D) -6 E) $\dfrac{3}{2}$

6) What is the solution set for the inequality $2x^2 + x < 6$?

A) $-2 < x < \dfrac{3}{2}$ B) $x > \dfrac{3}{2}$ or $x < -2$ C) $x < \dfrac{3}{2}$

D) $\dfrac{3}{2} < x < 2$ E) $x < -2$

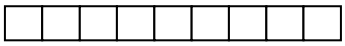
7) The parabola $y = x^2 - bx + 4$ has its vertex on the x-axis. What are the values of b?

A) ± 4 B) ± 5 C) ± 6 D) ± 7 E) ± 8

8) Let the function $f(x)=\dfrac{1}{x-1}$. Find $f(f(f(x)))$.

A) $\dfrac{x-1}{2-x}$ B) $\dfrac{x-2}{x-3}$ C) $\dfrac{2-x}{2x-3}$ D) $\dfrac{x-3}{2-x}$ E) $\dfrac{2-x^2}{x^2-3}$

9) Each of the 9 squares shown contains one number chosen from 1, 2, 3, 4, 5, 6, 7, 8, and 9. No number is repeated. Suppose that the sum of the numbers in the first five squares is 35 and that the sum of the numbers in last five squares is 16. What number goes to the fifth square?



A) 4 B) 5 C) 6 D) 7 E) 8

10) How many diagonals does a regular dodecagon have? A dodecagon is a polygon with twelve vertices.

A) 108 B) 132 C) 144 D) 84 E) 54

11) If $ab>0$ and $ac>0$, then the line $ax+by+c=0$ can not pass through which of the following quadrants?

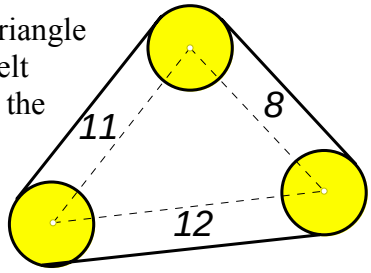
A) Quadrant I B) Quadrant II C) Quadrant III

D) Quadrant IV E) Quadrant I and Quadrant II

12) If the roots of $x^3+kx^2+hx-2009=0$ are $a,b,$ and $c,$ find $a^2+b^2+c^2$.

A) k^2-2h B) $k-2009$ C) k^2-4018 D) $\dfrac{k}{h}$ E) $2009-h$

13) * Circles of radius 2 are centered at the vertices of a triangle with side lengths 8, 11, and 12. Find the length of a belt that fits tightly around those three circles as shown in the figure.



A) 18.43 B) $31+4\pi$ C) $31+6\pi$

D) $31+\sqrt{3}\pi$ E) $31+\sqrt{2}\pi$

14) $n = 2009^2 - 2008^2 + 2007^2 - 2006^2 + \cdots + 5^2 - 4^2 + 3^2 - 2^2$ is not divisible by which of the following?

A) 4 B) 251 C) 1004 D) 2011 E) 512

15) * Suppose that $A \neq 0$ and $Ax^3 + Ax^2 + Bx + C = 0$ for $x = \pm 1$. Which of the following is a root of multiplicity 2 of the equation?

A) 1 B) -1 C) 0 D) 2 E) -2

16) * The number M has 1111 digits, each of which is equal to 1, $M = 111\ldots111$. What is the sum of the digits of $1001 \times M$?

A) 1111 B) 1221 C) 1122 D) 2112 E) 2222

17) * What are the last 3 digits of the number 3^{98} ?

A) 998 B) 899 C) 999 D) 989 E) 889

18) What is the coefficient of a^{-1003} in the binomial expansion of $\left(a - \frac{1}{\sqrt{a}}\right)^{2009}$?

A) 2009 B) -2009 C) 2017036 D) -2017036 E) 2008

19) For $a, b > 1$, when simplified, what does the expression $(\log b)(\log_b a) \div \log\left(\frac{1}{a}\right)$ become?

A) 1 B) -1 C) $-\frac{1}{2}$ D) $\frac{1}{2}$ E) 2

20) What is the range of the function $y = \left(\frac{\cos^{-1}(3x-1)}{\pi} + 1\right)^2$ with the domain $[0, 2/3]$?

A $[0, \infty)$ B) $[1, 4]$ C) $(1, 4)$ D) $(1, 3)$ E) $[0, \pi^2]$

21) If $8 \tan x = 3 \cos x$, what is the value of $\sin x$?

A) -3 B) $\frac{1}{3}$ C) $-\frac{1}{3}$ D) -1 E) $\frac{3}{8}$

22) Write the polar equation $r = \csc\left(\theta + \frac{\pi}{3}\right)$ in rectangular form with y expressed as a function of x.

A) $y = \sqrt{3}x - 2$ B) $y = \sqrt{3}x + 2$ C) $y = -\sqrt{3}x - 2$

D) $y = -\sqrt{3}x + 2$ E) None of these

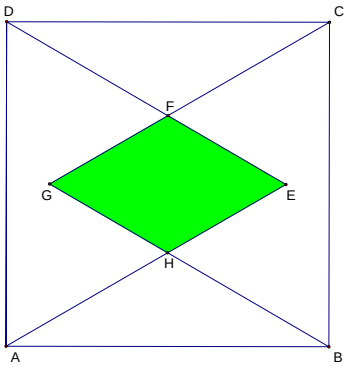
23) * Let $A = \sqrt{x + \sqrt{6x - 9}}$ and $B = \sqrt{x - \sqrt{6x - 9}}$. Find the real value(s) of x so that $A + B = \sqrt{6}$, and A and B are real numbers.

A) 3 B) 0, 3 C) $\left[\frac{3}{2}, 3\right]$ D) $(-\infty, 3]$ E) $[3, \infty)$

24) Two planes left an airport at the same time. One flew 200 mph at a bearing $N30^\circ E$ (30° east of north) and other at 210 mph at a bearing $E20^\circ S$ (20° south of east). To the nearest mile, how far apart are they after 2 hours?

A) 500 miles B) 527 miles C) 528 miles D) 529 miles E) 530 miles

25) Two equilateral triangles ΔADE and ΔBGC are inside a square $ABCD$ as shown in the figure. Find the ratio of the shaded area to the area of the square.



A) $\frac{7\sqrt{3}}{6} - 2$ B) $\frac{7\sqrt{3}}{8} - \frac{3}{2}$ C) $\frac{5\sqrt{3}}{3} - 1$

D) $\frac{2\sqrt{3}}{3} - \frac{1}{2}$ E) $\frac{2\sqrt{3}}{3} - 1$

26) Three circles of the same radius 2 intersect at a point O and each two intersect at A, B, and C respectively. What is the radius of the circle that circumscribes ΔABC?

A) 1 B) 1.5 C) 2 D) 2.5 E) 3

27) * If a and b are randomly chosen integers from -3 to 3 inclusive, what is the probability that $x^2 + ax + b = 0$ has real solutions?

A) $\frac{13}{49}$ B) $\frac{17}{49}$ C) $\frac{28}{49}$ D) $\frac{31}{49}$ E) $\frac{34}{49}$

28) * The circle $(x-6)^2+(y-2)^2=64$ lies in all four quadrants. I_1 is the area of the portion of the circle's interior that is in Quadrant I, I_2 is the area of the portion of the circle's interior that lies in Quadrant II, I_3, in Quadrant III, and I_4, in Quadrant IV. Find $I_1-I_2+I_3-I_4$.

A) $48(\sqrt{7}-1)$ B) 24 C) $30\sqrt{3}-10$ D) 48 E) $64\pi-30$

29) * If x and y are positive integers and $2xy=2009-3y,$ then how many positive solutions (x,y) are there?

A) 1 B) 2 C) 3 D) 4 E) 5

30) An arbitrary triangle of perimeter 10 is formed. A second triangle is formed by joining the midpoints of the first triangle. A third triangle is formed by joining the midpoints of the second triangle, and so on indefinitely. Find the total length of all line segments in the resulting configuration.

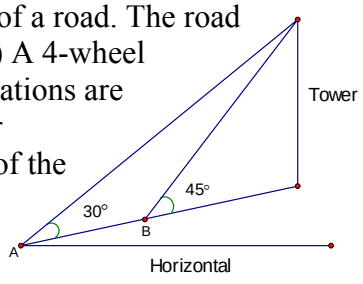
A) 15 B) 20 C) 30 D) 100 E) ∞

31) What is the value of the following product?

$$\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2007}}\right)\cos\left(\frac{\pi}{2^{2006}}\right)\cdots\cos\left(\frac{\pi}{2^{3}}\right)\cos\left(\frac{\pi}{2^{2}}\right)$$

A) $\frac{1}{2^{2010}}$ B) $\frac{1}{2^{2009}}$ C) $\frac{1}{2^{2008}}$ D) 1 E) 0

32) A 750 foot tall vertical tower is located at the top of a road. The road has a constant inclination. (See the diagram right.) A 4-wheel drive vehicle is driven up the road and two observations are made. At the first observation point (A), the tower subtends an angle of 30 degrees from the surface of the road. The second observation at point (B) is 500 feet further along the road and the tower subtends an angle of 45 degrees. Determine the inclination of the road.



A) 20° B) 24° C) 24.4° D) 25° E) 15°

33) Suppose that $f(x) = ax^2 + bx + a$ satisfies the equation $f\left(x + \frac{7}{4}\right) = f\left(\frac{7}{4} - x\right)$ and that the equation $f(x) = 7x + a$ has only one solution. What is the value of $a + b$?

A) 3 B) 4 C) 5 D) 6 E) 7

34) * How many pairs of positive integers (a, b) with $a + 2b < 80$ satisfy the equation $a + \frac{1}{b} = 7\left(b + \frac{1}{a}\right)$?

A) 7 B) 8 C) 9 D) 10 E) 11

35) For every pair of positive integers a and b, we consider the operation "#" on positive integers with the following three properties:

a) $a \# a = a + 2$
b) $a \# b = b \# a$
c) $\frac{a \#(a + b)}{a \# b} = \frac{a + b}{b}$

What is $3 \# 7$?

A) 7 B) 21 C) 49 D) 63 E) 77

36) * How many distinct pairs of integers (x, y) satisfy the equation $x^2 + y^2 = 2009$?

A) 2 B) 4 C) 6 D) 8 E) 10

37) A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

A) 3 B) 4 C) 5 D) 6 E) 7

38) * Two friends agree to meet at library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

A) $\frac{3}{7}$ B) $\frac{4}{5}$ C) $\frac{4}{9}$ D) $\frac{5}{9}$ E) $\frac{7}{9}$

39) * If $\tan x = 3\tan y$ for $0 \le y < x < \frac{\pi}{2}$, what is the maximum value of $x - y$?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{12}$ E) $\frac{5\pi}{12}$

40) Sally walked 4 miles north, then 2 miles east, 1 mile south, $\frac{1}{2}$ mile west, $\frac{1}{4}$ mile north, and so on. If she continued this pattern indefinitely, how far from her initial point will she approach?

A) 3 miles B) 3.2 miles C) 4 miles D) $\frac{13\sqrt{5}}{8}$ miles E) $\frac{8\sqrt{5}}{5}$ miles

41) * The number $2008ab2009$ is divisible by 99, where a and b denote two missing digits in base 10. What is the value of $(a^2+1)(b^2+1)+2a+3b$?

A) 98 B) 99 C) 100 D) 101 E) 102

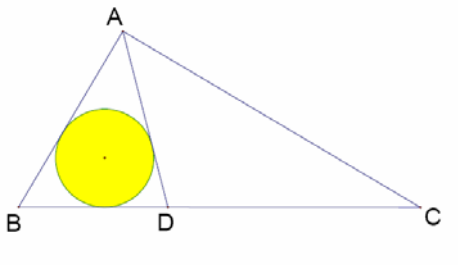
42) The altitudes of a triangle are 210, 195 and 182. What is the area of the triangle?

A) 22183 B) 22183.25 C) 22182 D) 22181 E) 22181.25

43) In a right triangle ΔABC with legs $AB = 3$ and $AC = 4$, the angle bisector of $\angle BAC$ is constructed so that D is on $\overline{BC}$. Find the radius of the circle inscribed to the triangle ΔABD. (See the accompanying figure.)

A) $\frac{9-3\sqrt{2}}{7}$ B) $\frac{7-3\sqrt{2}}{9}$ C) $\frac{9-2\sqrt{3}}{7}$

D) $\frac{6-3\sqrt{2}}{5}$ E) $\frac{3\sqrt{2-\sqrt{2}}}{4}$



44) Let $r_1, r_2, \ldots, r_n$ be n positive integers, not necessarily distinct, such that

$$(x+r_1)(x+r_2)\cdots(x+r_n) = 2009 = x^n + 56x^{n-1} + \cdots + 2009.$$

What is the value of n?

A) 4 B) 20 C) 73 D) 758 E) 2009

45) * Let $f(x)$ be an even function on $(-\infty,\infty)$ and $f(x+2)=f(x-2)$. If $f(x)=0$ has only three real roots in $[0,4]$ and one of them is 4, find the number of real roots of $f(x)$ in $(-8,10]$.

A) 6 B) 7 C) 8 D) 9 E) 10

46) In the table below we write all the different products of two distinct counting numbers between 1 and 100:

$$\begin{array}{cccccc} 1\cdot 2, & 1\cdot 3 & \dots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \dots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Find the sum of all these products.

A) 300,000 B) 450,000 C) 640,120 D) 12,582,075 E) 25,164,150

47) * Suppose that the half circle $y=1+\sqrt{4-x^2}$ and the line $y=kx+4-2k$ have two points of intersection. What is the range of the value k?

A) $[1,3]$ B) $(1,3)$ C) $\left(\frac{1}{2},\frac{3}{4}\right)$ D) $\left[\frac{5}{12},\frac{3}{4}\right]$ E) $\left(\frac{5}{12},\frac{3}{4}\right]$

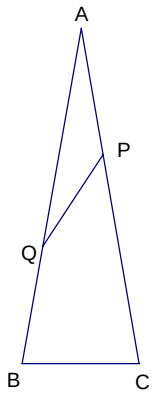
48) If $x>1,\ y>1$, for what value of x does the equation $x^y=y^x$ in y have only one solution?

A) e B) $\frac{e}{2}$ C) 1.6 D) 3 E) 3.5

49) Suppose that the function $f(x)=\log_C\frac{x-2}{x+2}$ defined for all x in an interval $[a,b]$ is decreasing. If the range of the function is $[\log_C C(b-1),\log_C C(a-1)]$ when $b>a>2$, find the values of C.

A) $0<C<\frac{1}{2}$ B) $0<C<\frac{1}{9}$ C) $C>1$ D) $\frac{1}{9}<C<1$ E) $\frac{1}{3}<C<\frac{1}{9}$

50) In a triangle ΔABC, $m\angle ABC = m\angle BCA$ and the points P and Q are located on $\overline{AC}$ and $\overline{AB}$ as shown in the figure so that $AP = PQ = QB = BC$. Find $m\angle BAC$.



A) 15° B) 16° C) 17°

D) 20° E) 21°