Twenty-ninth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 1, 2003

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Solutions

1. $\dfrac{3^{2002} \cdot 7^{2004}}{21^{2003}} = \dfrac{3^{2002} \cdot 7^{2004}}{3^{2003} 7^{2003}} = \dfrac{7^{2004-2003}}{3^{2003-2002}} = \dfrac{7}{3}$ so the answer is (E).

2. $\dfrac{\sqrt[6]{a^3 b^2}}{\sqrt[8]{a^5 b^3}} = \dfrac{1}{a^{\frac{5}{8}-\frac{3}{6}} b^{\frac{3}{8}-\frac{2}{6}}} = \dfrac{1}{a^{\frac{1}{8}} b^{\frac{1}{24}}} = \dfrac{1}{\sqrt[8]{a}\,\sqrt[24]{b}}$ so the correct answer is (D).

3. If the number in question is "x" then Mary calculated $11x + 7$ and obtained 40. So $x = (40 - 7)/11$ or $x = 3$. Mary was supposed to do $7x + 11$ which is 32. Answer: (D).

4. The slope of the line $6x + 9y + 5 = 0$ is clearly $-\dfrac{6}{9}$ or $-\dfrac{2}{3}$. Hence the slope of a line perpendicular to it is $\dfrac{3}{2}$. Hence the answer is (C).

5. Factoring $2x + 1$ in the left hand side we get $(2x - 3)(x + 1) + (2x - 3)(2x + 1) = (2x - 3)(x + 1 + 2x + 1) = (2x - 3)(3x + 2)$. So the roots are $\dfrac{3}{2}$ and $-\dfrac{2}{3}$, so their product is -1. Answer: (B).

6. Bringing the two fractions to the same common denominator we get $\dfrac{5x^2 - 3x + 2}{x^2 - 1} = 5$. If $x \neq \pm 1$ then we have equivalently $5x^2 - 3x + 2 = 5x^2 - 5$ which leads to a linear equation whose solution is $x = \dfrac{7}{3}$. Answer: (D).

7. Since $x^2 - 6x + 9 = (x-3)^2$ we have equivalently $|x-3| = 4$ (notice that for a real number a we have $\sqrt{a^2} = |a|$). Hence $x_{1,2} = 3 \pm 4$ and since we are looking for positive solutions, $x = 7$ is the only one. Answer: C.

8. Adding the two equations together we get $3x + 3y = 2$. Hence $x + y = \frac{2}{3}$. Answer: (E).

9. If we denote the two digits by a and b ($a \neq 0$) we get the equation $10a + b = 2(a+b)$ or equivalently $b = 8a$ which has only one solution $a = 1$ and $b = 8$. Since $ab = 8$ the correct answer is (A).

10. The volume of a parallelepiped with dimensions L, W and H is LWH. Since $L = \frac{18}{3} = 6\ yards$, $W = \frac{9}{3} = 3\ yards$ and $H = \frac{8}{12}\ ft = \frac{\frac{2}{3}}{3} = \frac{2}{9}\ yards$. Then the volume in cubic yards is $6 \times 3 \times \frac{2}{9} = 4$. Answer: (B)

11. Clearly $(3,2,1) = \frac{2\times 3\times 2 + 3\times 2\times 1 + 4\times 3\times 1}{(3+2+1)^2} = \frac{12+6+12}{36} = \frac{5}{6}$, so the best choice here is (D).

12. Expanding the left hand side we obtain $(a^2 + b^2 + c^2 - ab - ac - bc)(a+b+c) = a^3 + a^2b + a^2c + b^2a + b^3 + b^2c + c^2a + c^2b + c^3 - a^2c - ab^2 - abc - a^2c - abc - ac^2 - abc - b^2c - bc^2 = a^3 + b^3 + c^3 - 3abc$ and so the closest choice is (A).

13. The number of all these permutations including N is $4! = 1\times 2\times 3\times 4 = 24$. Each of the digits 1, 2, 3, or 4 appears on each position (units, tens, hundreds, thousands) 6 times (the number of permutations of three objects). So the sum is $1111\times 6\times(1+2+3+4) = 66660$. Answer:(D).

14. Let b be the base of the triangle and h be its altitude to this base. The new triangle has base $B = b + 0.\overline{6}b = b(1 + \frac{6}{9}) = \frac{5}{3}b$ and altitude $H = h - 0.4h = h(1 - \frac{2}{5}) = \frac{3}{5}h$. The area of the new triangle is $A = \frac{1}{2}BH = \frac{1}{2}\frac{5}{3}b\frac{3}{5}h = \frac{1}{2}bh$ which means there is no change in area. Hence the correct answer is (B).

15. If $x \neq -\frac{1}{3}$ or $\frac{m}{6}$ we have equivalently $(x+2)(6x-m) = (3x+1)(2x-1)$ or $(13-m)x = 2m-1$. Therefore if $m = 13$ there is no solution and so we need to chose (E). There are more solutions to this problem though not listed as answers. If $m \neq 13$ then the solution of the last equation is $\frac{2m-1}{13-m}$. Writing that $\frac{2m-1}{13-m} = -\frac{1}{3}$ or $\frac{m}{6}$ we get $m = -2$ or 3. In these cases the "solution" doesn't satisfy the existance condition and so the given equation has no solution.

16. Regrouping we get the equivalent equality $(4x^2-1)y + 4x^2 - 4x + 1 = 0$. Since this must be true for every value of y we should have $4x^2 - 1 = 4x^2 - 4x + 1 = 0$ which yields only the solution $x = \frac{1}{2}$. Answer: (D).

17. Using the change of base formula we obtain $\log_5 24 = \frac{\log_{10}(2^3 \times 3)}{\log_{10} 5} = \frac{3\log_{10} 2 + \log_{10} 3}{\log_{10} 10 - \log_{10} 2} = \frac{3a+b}{1-a}$. Answer: (C).

18. We have $\log_2(\log_3 x) = \log_3(\log_2 y) = 1$ which means that $\log_3 x = 2$ and $\log_2 y = 3$. Therefore $x = 2^3$, $y = 3^2$ and so $x + y = 8 + 9 = 17$. Answer: (D).

19. The sum of the solutions of a polynomial equation $ax^n + bx^{n-1} + ... = 0$ is $\frac{-b}{a}$. Based on this fact the sum of the other two solutions is $\frac{25}{6} - 2 = \frac{13}{6}$. Answer:(D).

20. The sum of the new roots is $3(r+s) + 2 = 3 \times \left(-\frac{4}{3}\right) + 2 = -2$ and their product $(3r+1)(3s+1) = 9rs + 3(r+s) + 1 = 9 \times \left(\frac{5}{3}\right) + 3 \times \left(-\frac{4}{3}\right) + 1 = 12$. Then an equation with these new roots will be $x^2 + 2x + 12 = 0$ which gives (D) as the correct choice.

21. The given sequence continues as follows $2, 2, 4, 6, 3, 2, 5, 0, 5, 5, 3, 1, 4, 5, 2, 0, 2, 2, ...$ and from here on the terms clearly repeat. One can count the period as being 16. Since $2003 = 125 \times 16 + 3$ the 2003-rd term is the same as the third term which is 4. Hence the answer is (E).

22. The area of the equilateral triangle is $\frac{\sqrt{3}x^2}{4}$ and the area of the square is y^2. Since these two expressions are equal we obtain $\frac{y}{x} = \frac{\sqrt[4]{3}}{2}$. Answer: (E).

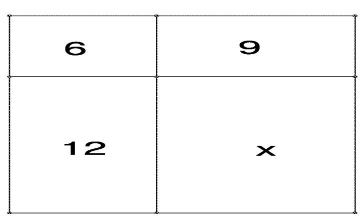
23. Let x be Wanda's annual income. Then the equation that x satifies is $\frac{(t + 0.25)x}{100} = \frac{t \times 30000}{100} + \frac{(t + 1)(x - 30000)}{100}$. In solving this equations t disappears and the solution is $x = 40000$ which says that the answer is (A).

24. A number is divisible by 7 and 13 if it is divisible by their product $(7 \times 13 = 91)$. Since $2003 = 91 \times 22 + 1$ there are 22 numbers between 1 and 2003 which are divisible by 91: 91, 91×2, 91×3, ..., 91×22. Of these, there are only three which are divisible by 49: 91×7, 91×14 and 91×21. Thus there are only 19 numbers satifying the required conditions. Answer: (C).

25. The given equality implies that $a^2 + b^2 = \sqrt{3}ab$ which after squaring both sides gives $a^4 - a^2b^2 + b^4 = 0$. Multiplying both sides of this resulting equality by $a^2 + b^2$ we get $a^6 + b^6 = 0$. Therefore the answer is (E).

26. Since $2003 = ac^2 + b^2 > a^3$ it follows that $a < 2003^{1/3}$ which means $a \leq 12$. Hence $c = a + 2 \leq 14$. On the other hand $2003 < c^3 + c^2$ which implies $c \geq 13$ because $12^3 + 12^2 = 1872 < 2003$. Therefore we need to try only two possibilities $c = 13$ or $c = 14$. After we check that $c = 14$ does not work and $2003 = 11 \times 13^2 + 12^2$ we calculate $2a + 4b - 5c = 2 \times 11 + 4 \times 12 - 5 \times 13 = 5$. Answer: (B).

27. The area of a rectangle of width W and length L is equal to WL. Therefore $\frac{6}{12} = \frac{9}{x}$ and so $x = 18$. Answer: (A).



28. Let x be the length of John's strip in feet. Then Mary's is going to mow a reactangular region with dimensions $72 - 2x$ and $65 - 2x$. Since the area of this reagion should be $\frac{1}{2}(72 \times 65)\ ft^2$, we get the quadratic equation $(72 - 2x)(65 - 2x) = 2340$. This is equivalent with $2x^2 - 137x + 1170 = 0$. The discriminant of this is $\Delta = 137^2 - 4 \times 2 \times 1170 = 9409 = 97^2$. Hence we get two solutions $x_1 = (137 + 97)/4 = 58.5\ ft$ and

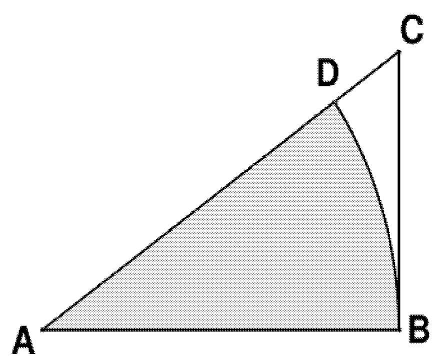
$x_2 = (137 - 97)/4 = 10\ ft$. Clearly the first solution is not feasible and so the correct choice here is (B).

29. We have $x^2 - y^2 - z^2 - 2yz + x + y + z = x^2 - (y+z)^2 + (x+y+z) = (x+y+z)(x-y-z) + (x+y+z) = (x+y+z)(x-y-z+1)$. Hence $x-y-z+1$ is the factor we are looking for and the answer is (E).

30. It is easy to factor the rule of h as follows $h(x) = (3x-1)(2x-1)$. We see that $h(1) = 2$ which is a prime. If $x \geq 2$ then $3x - 1 > 2x - 1 \geq 3$ and so $h(x)$ would not be prime. $h(0) = 1$ is not considered a prime number. For $x \leq -1$ then $1 - 3x > 1 - 2x \geq 3$ and then again $h(x)$ is composite. Therefore there is only one prime value of h. Answer: (C).

31. Let us denote by D the intersection of the angle bisector of the angle $\angle A$ and the altitude from B. Also let E be the point on the side $\overline{AC}$ where the altitude from B intersects it. The triangle $\triangle ADE$ is a right tringle and $m(\angle ADE) = 180^\circ - 120^\circ = 60^\circ$. Hence $m(\angle DAE) = 90^\circ - 60^\circ = 30^\circ$. Thus $m(\angle A) = 2 \times 30^\circ = 60^\circ$. This means that the triangle $\triangle ABC$ is equilateral. Then its area is $\frac{2^2\sqrt{3}}{4} = \sqrt{3}\ cm^2$. Answer: (A).

32. We have $\tan(A) = \frac{3}{4}$ and so $m(\angle A) = \arctan(\frac{3}{4})$. Then the area of the shaded sector ABD is equal to $\frac{1}{2}m(\angle A)4^2 = 8 \times \arctan(\frac{3}{4}) \approx 5.148008870$ and so the best choice here is (D).



33. We can write $g(x) = \sqrt{5}(\frac{1}{\sqrt{5}}\sin x + \frac{2}{\sqrt{5}}\cos x) = \sqrt{5}\sin(x+\alpha)$ where $\alpha = arctan\ 2$ (Notice that then $\tan\alpha = 2$, $\cos\alpha = \frac{1}{\sqrt{1+tan^2\alpha}} = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{2}{\sqrt{5}}$). Therefore $|g(x)| \leq \sqrt{5}$ for all angles x. Since $\frac{9^2}{4^2} > 5$ it follows that $\frac{9}{4} > \sqrt{5}$. Hence the answer must be (A).

34. Denote by x the unkown of this problem. Then x should satisfy a simple linear equation: $0.4 \times x + 0.15 \times (30 - x) = 0.2 \times 30$. One solves this equation and finds $x = 6$ liters. Answer: (C).

35. Let A be the total number of occupied houses. Clearly $0.6A$ houses contained only one person. Of these $0.75 \times 0.6A$ contained exactly one female or, equivalently $0.45A$. This means that the percentage we are looking for is 45% and the answer is (C).

36. Denote the number 2^{2003} by N. Then N has k digits if and only if $10^{k-1} \leq N < 10^k$. Equivalently, $k - 1 \leq log_{10}N < k$ or $k - 1 \leq 2003 \times log_{10}2 < k$. Using a calculator $2003 \times log_{10}2 \approx 602.9630814$ which means that $k = 603$. Answer: (D).

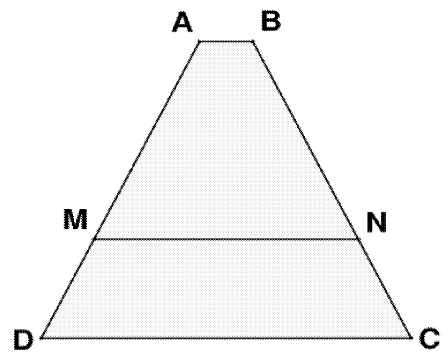
37. The relation given implies that $b - a = ab$. Hence $\dfrac{a - 2ab - b}{2a + 3ab - 2b} = \dfrac{-2ab - ab}{3ab - 2ab} = \dfrac{-3ab}{ab} = -3$. Answer: (B).

38. Applying the law of sines we get $\dfrac{3}{\sin C} = \dfrac{\sqrt{6}}{\sin 45^\circ}$. Solving for $\sin C$ we obtain $\sin C = \dfrac{\sqrt{3}}{2}$. Since $\angle C$ is an acute angle we get that $C = 60^\circ$. Therefore $m(\angle BAC) = 180^\circ - 45^\circ - 60^\circ = 75^\circ$. Answer: (B).

39. $*$ If we let $x = \frac{1}{y}$ in the given equality we get $f(1)^2 = \frac{1}{y} f(y)^2$ for all positive y. This implies that $f(y) = f(1)\sqrt{y}$ for all y. Because $f(2) = 6$ we get $f(1) = \dfrac{6}{\sqrt{2}} = 3\sqrt{2}$ and so $f(y) = 3\sqrt{2y}$ for all y. Hence $f(50) = 3\sqrt{2 \times 50} = 30$. Answer: (B).

40. We have the formula $a^3+b^3=(a+b)(a^2-ab+b^2)$. Then $a^3+b^3=85-ab$. On the other hand $ab=\left[(a+b)^2-(a^2+b^2)\right]/2=(1-85)/2=-42$. Thus $a^3+b^3=85-(-42)=127$. Answer: (A).

41. $*$ Let E and F be the projections of A and B on the base $\overline{DC}$. Also, denote by G and H the intersections of $\overline{MN}$ with $\overline{AE}$ and $\overline{BF}$ respectively. It is easy to see that $DE=FC=3\ cm$. Then $AE=4\ cm$. Hence the area of the trapezoid $ABCD$ is $\frac{1}{2}(7+1)\times 4=16\ cm^2$. Thus if we set $HN=x$, from the similarity of the triangles $\triangle BHN$ and $\triangle BFC$ we get $\frac{BH}{4}=\frac{x}{3}$. This gives $BH=\frac{4x}{3}$ and the area of $ABMN$ can then be calculated in terms of x: $\frac{1}{2}\left[(2x+1)+1\right]\frac{4x}{3}=8$. Solving for x we obtain $x=2$. Therefore $MN=5$. Answer: (D).



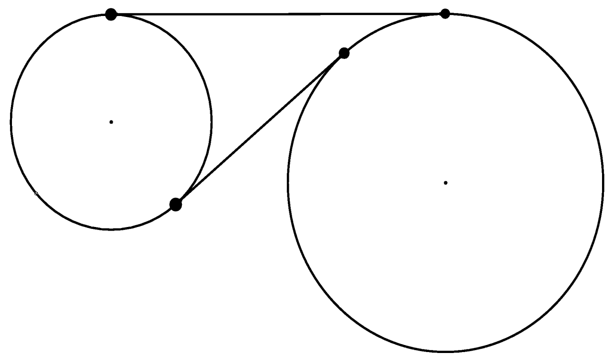
42. The formula for the sum of an arithmetic progression $a_1, a_2, ..., a_n$ is $S=\dfrac{1}{2}(a_1+a_n)n$. In this case $n=(52-7)/3+1=16$. Then $S=\frac{1}{2}(7+52)\times 16=472$. Answer: (E).

43. Since $x\to\tan x$ is an odd periodic function with period $180°$ we get $\tan(360°-A)=-\tan A=-\dfrac{\sin A}{\cos A}=-\dfrac{\frac{3}{5}}{\frac{4}{5}}=-\dfrac{3}{4}=-0.75$. Answer: (E).

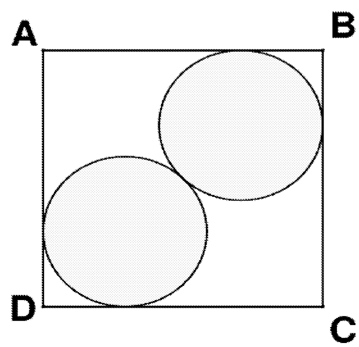
44. There are a total of $\binom{10}{4}=\frac{10\times 9\times 8\times 7}{1\times 2\times 3\times 4}=210$ possible ways to seat 4 people on 10 seats.

Of these, only 7 are as required. Hence the probability is $\frac{7}{\binom{10}{4}} = \frac{1}{30}$. Answer: (D).

45. $*$ Let R and r be the radii of two circles and d the distance between their centers. Applying the Pythagorean Theorem we get that Length of Big Tangent Segment= $\sqrt{d^2 - (R-r)^2}$ and Length of Small Tangent Segment= $\sqrt{d^2 - (R+r)^2}$. So the first segment has length $\sqrt{65^2 - 16^2} = 63$ and the second has length $\sqrt{65^2 - 56^2} = 33$. Then their sum is 96 cm and so the answer is (A).

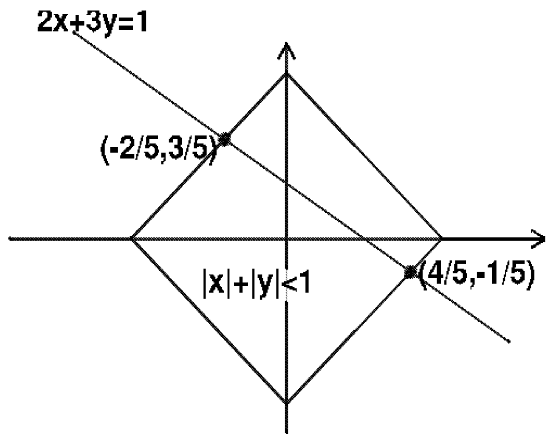


46. $*$ In a square of sides L the diagonal is equal to $L\sqrt{2}$. The diagonal of the square in our case is equal to $r\sqrt{2} + 2r + r\sqrt{2} = 2\sqrt{2}$. Solving for r will give $r = 2 - \sqrt{2}$. Answer: (E).

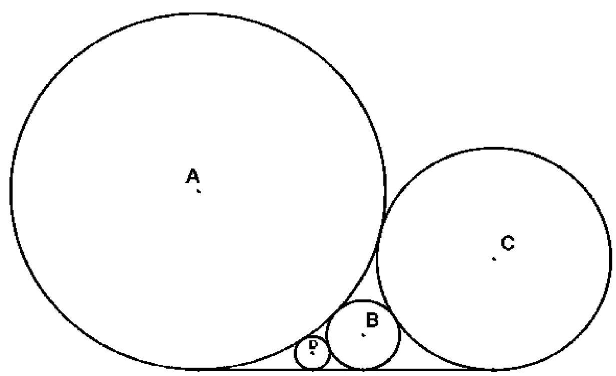


47. Let $2n+1$ ($n \geq 1$) be the number of these consecutive integers. If k is the smallest of them then $k+(k+1)+...+(k+2n) = k(2n+1)+2n(2n+1)/2 = (2n+1)(k+n) = 2003$. But 2003 is a prime number. Hence $2n+1 = 2003$ and so $n = 1001$ and $k = -1000$. Answer: (D).

48. $*$ Intuitively, the probability of choosing a point from a set is directly proportional to its area (and is independent of its shape). In this case, the probability will be the ratio of the area of the trapezoidal region $2x+3y<1$ to the area of the square $|x|+|y|<1$. Calculating the bases of the trapezoidal region one obtains $\frac{3}{5}\sqrt{2}$ and $\frac{4}{5}\sqrt{2}$. The height of the trapezoid is $\sqrt{2}$ and so its area is $\frac{7}{5}$. Thus the required probability is $\frac{7}{10}$.Answer:(A).



49. $*$ As in Problem 45 if two circles are tangent to each other then the distance between their centers is $R+r$ and the length of the big tangent segment is equal to $\sqrt{(R+r)^2-(R-r)^2} = 2\sqrt{Rr}$. Applying this formula for the pairs of circles $\{C(A,9), C(D,r_2)\}$, $\{C(D,r_2), C(B,1)\}$ and $\{C(A,9), C(B,1)\}$ we obtain $2\sqrt{9r_2}+2\sqrt{r_2} = 2\sqrt{9}$. Hence $r_2 = \frac{9}{16}$. Similarly, one gets $r_1 = \frac{9}{4}$.Then $\frac{r_1}{r_2} = 4$. Answer: (D).



50. If n is a positive integer then $x = \frac{2n}{n^2+1}$ and $y = \frac{n^2-1}{n^2+1}$ satisfy the equation $x^2+y^2=1$. These formulae give infinitly many values of x and so the correct answer is (E).