Tennis Balls in a Can

Common Core State Standards

Visualize relationships between two-dimensional and three-dimensional objects.

MGSE9-12.G.GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

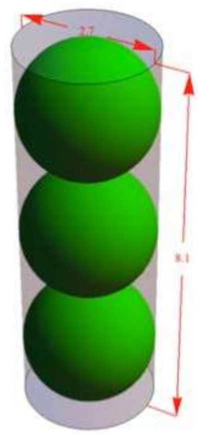
This task is inspired by the derivation of the volume formula for the sphere. If a sphere of radius 1 is enclosed in a cylinder of radius 1 and height 2, then the volume not occupied by the sphere is equal to the volume of a "double-napped cone" with vertex at the center of the sphere and bases equal to the bases of the cylinder. This can be seen by slicing the figure parallel to the base of the cylinder and noting the areas of the annular slices consisting of portions of the volume that are *inside* the cylinder but *outside* the sphere are the same as the areas of the slices of the double-napped cone (and applying Cavalieri's Principle). This almost magical fact about slices is a manifestation of Pythagorean Theorem. The visualization required here is used in calculus, in connection with procedures for calculating volumes by various slicing procedures.

Materials

- Pencil
- Handout

Tennis Balls in a Can

The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and $3 \times 2.7 = 8.1$ inches high.

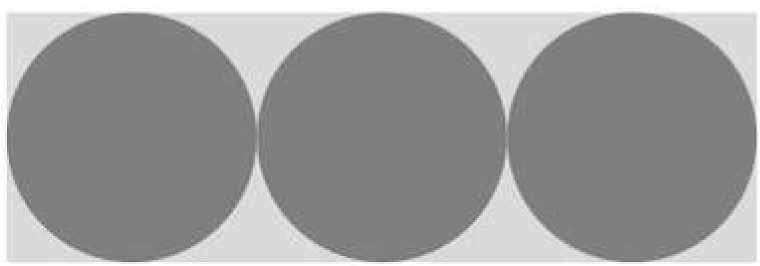


(a) Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? With what dimensions?

The shadow is a rectangle measuring 2.7 inches by 8.1 inches.

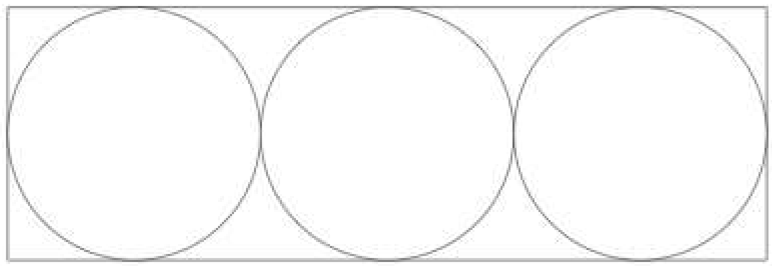
(b) If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?

The shadow is a light rectangle (2.7 × 8.1 inches) with three disks inside. It looks like a traffic light:



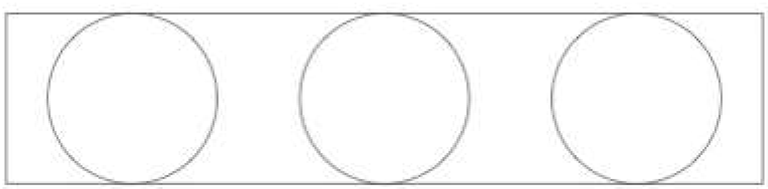
(c) The *central axis* of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a *cross section*. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

The image is similar to the previous one, but now only the outlines are seen:



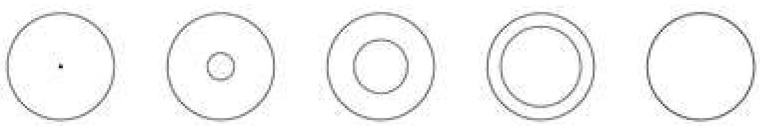
(d) If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

The intersection with the container is a narrower rectangle. The intersections with the balls are smaller circles. Because each ball touches the container along its whole "equator," the circles must touch the long sides of the rectangle:



(e) If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

The intersections are two concentric circles, except when $w = 0, 2.7, 5.4, 8.1$ and when $w = 1.35, 4.05, 6.75$. In the former case, we see a circle (from the container) and a point (where the plane touches a sphere). In the latter case, we see a single circle corresponding to a place where the equator of a ball touches the container.



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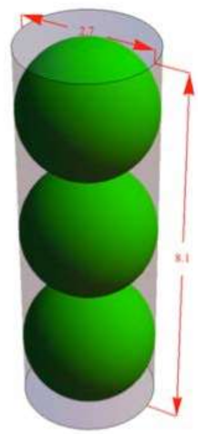

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