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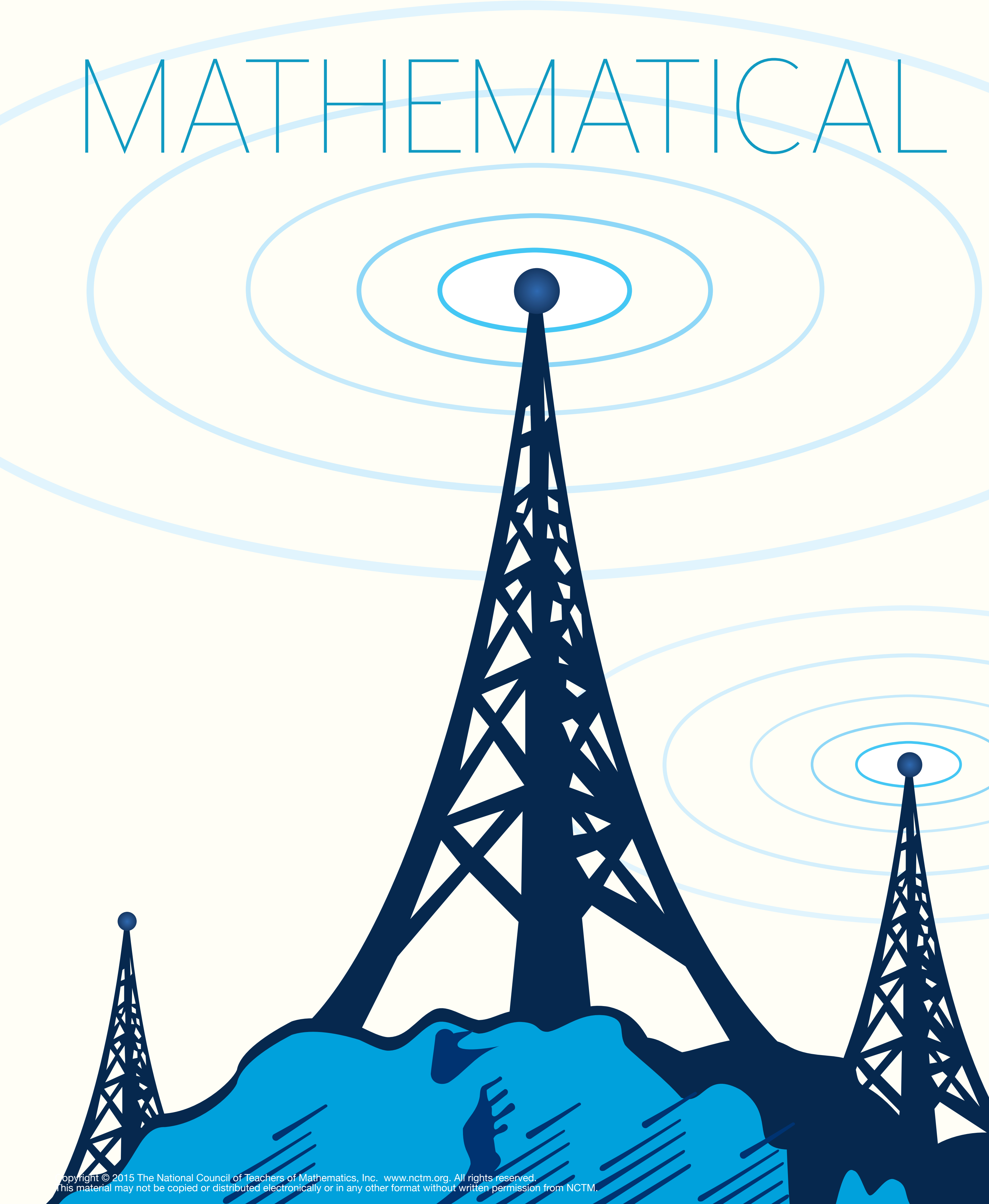

MODELING: A STRUCTURED PROCESS

A task about locating a lost cell phone illustrates the Common Core elements of mathematical modeling. An assessment rubric helps teachers evaluate student work critically.

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Mathematical modeling, in which students use mathematics to explain or interpret physical, social, or scientific phenomena, is an essential component of the high school curriculum. The Common Core State Standards for Mathematics (CCSSM) classify modeling as a K–12 standard for mathematical practice and as a conceptual category for high school (CCSSI 2010, p. 7). CCSSM also describes mathematical modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (CCSSI 2010, p. 72). The main goal of this article is to elaborate on the process of modeling as described by CCSSM, paying particular attention to the modeling elements. We highlight the practical aspects of modeling through a concrete example, carefully analyzing to make the process of mathematical modeling more accessible to teachers and students.

One way to foster students' proficiency in modeling is to consider explicit use of the modeling elements. Having students understand the modeling process is analogous to having students be familiar with the scientific process as they are conducting an experiment. The example that we provide shows how a single activity can target multiple concepts and approaches, allowing students to understand those concepts and make connections among them. Students' work should be evaluated not only on the outcome of their model but also on the thought process demonstrated in more than one iteration of the modeling cycle. To help with this understanding, we also provide a useful evaluation rubric with criteria for assessing student work in modeling.

ELEMENTS OF MODELING

Understanding the elements that modeling problems contain promotes modeling as a mathematical practice. The modeling process depicted in **figure 1** is adapted from CCSSM. The elements, necessarily in this order, make up the six stages of the modeling cycle. The arrows in the diagram indicate the general path from beginning to end. After stage 5, "Validate conclusions," a decision must be made as to whether the model will need improvement; this decision determines whether we report the answer or cycle back to formulating a new or modified model (stage 2). Navigating through the modeling cycle may also include revisiting a past stage, such as correcting computations (stage 3) during the interpretation of solutions (stage 4), although this is not explicitly shown in the diagram.

The value in the modeling activity is a combination of the problem itself and the way it is managed in the classroom by allowing the students to recognize the modeling elements in their own work. **Table 1** provides an explicit description of the work that is connected to each element of the modeling cycle.

One hallmark of mathematical modeling is the decision-making process that students go through when formulating a model. Making assumptions is particularly important because it requires students to take assertive measures with confidence that their decisions will help them develop a reasonable model. It is important to distinguish between assumptions, which affect the model directly, and procedural choices (such as conducting the computations in units of inches versus centimeters), which are necessary but do not define the model. During the process of making assumptions, it is helpful to engage the students in a discussion about the effect of their assumptions

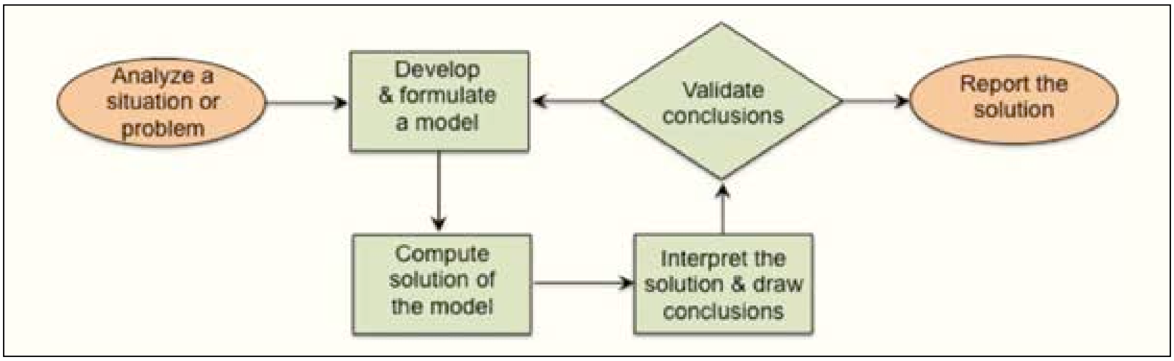


Fig. 1 Element titles within the modeling cycle (CCSSI 2010, p. 72) have been expanded to be more descriptive.

Table 1 Elements of Mathematical Modeling and Description of the Work That They Entail

Modeling Element	What It Entails
1. Analyze the situation or problem	• Identify a problem taken from an external context (often from an everyday life context) that must be solved or a situation that must be understood and explained. • Do background research if necessary. • Make sense of the situation or problem and understand the question.
2. Develop and formulate a model	• Determine all given information. • Determine what assumptions are necessary. • Translate the information given in the problem together with the assumptions into a mathematical problem that can be solved. • Use mathematics appropriate for the information given and assumed as well as the students' expertise.
3. Compute a solution of the model	• Solve the mathematical problem stated in the model. • Analyze and perform operations in the model. • Check for correctness.
4. Interpret the solution and draw conclusions	• Interpret the mathematical solution in terms of the original situation. • Draw conclusions that the solution implies about the original situation.
5. Validate conclusions	• Reflect on whether the mathematical answer makes sense in terms of the original situation (e.g., is the answer within a valid range of values?). • If the conclusions are satisfactory with regard to the accuracy needed, report the solution. If the conclusions are not satisfactory or need to be improved, go back to stage 2 ("Develop and formulate a model").
6. Develop and formulate a new or modified model	• Revise the assumptions made according to what was learned in the first solution and translate them into a new or modified mathematical problem that can be solved. • The type of mathematics in the current model may be different from the previous one. • Go through these stages again: Compute, Interpret, and Validate.
Report the solution	• Share your conclusions and the reasoning behind them.

and justification of their choices (Anhalt 2014). Consider the following example of a modeling problem, including how the elements of the cycle present themselves within the solution.

THE MODELING CYCLE IN CONTEXT: THE LOST CELL PHONE PROBLEM

To provide a concrete illustration of the modeling process, we discuss a specific modeling problem in a context that most students will relate to—a lost cell phone (see **fig. 2**). We offer a possible solution along with commentary that emphasizes the flow through the cycle.

Analyze the Situation or Problem

The problem provides three tower locations and the distances recorded by the towers from a cell phone signal; the students are asked to determine the

THE LOST CELL PHONE PROBLEM

A lost cell phone needs to be found. Fortunately, three cell phone towers detect the signal. A coordinate system used by the city indicates that the cell towers are located at (x, y) coordinates, measured in meters from the center of town:

- Cell tower 1 is at position (1200, 200).
- Cell tower 2 is at position (800, –450).
- Cell tower 3 is at position (–100, 230).

Tower 1 detects the signal at a distance of 1072.7 meters. Tower 2 detects the signal at a distance of 1213.7 meters. Tower 3 detects the signal at a distance of 576.6 meters. Create an approach for finding the location of the cell phone. Explain your reasoning.

Fig. 2 The problem specifies tower locations.

location of the cell phone. We encourage students who are not familiar with cell phone towers to look up pictures and basic information about them online. Some familiarity with the function of cell phone towers can be helpful in developing a model. The context should trigger a discussion of the desired accuracy in the solution.

Develop and Formulate a Model

Students can be prompted to consider what information they need to determine the location of the cell phone and whether all information is provided. The coordinates given in the problem suggest that the problem can be modeled in the *xy*-plane (in two dimensions). Specific assumptions could be that the distances from the towers to the cell phone are horizontal, that the distances recorded by the towers are accurate (i.e., they contain no errors), and that the cell phone is not moving.

On the basis of these assumptions, we can draw circles centered at the tower locations with radii equal to the distances to the cell phone recorded by the towers. This geometric approach can be represented algebraically by letting (x, y) denote the cell phone location and solving the following system of equations, which constitute the model:

$$(x - 1200)^2 + (y - 200)^2 = 1072.7^2$$
$$(x - 800)^2 + (y + 450)^2 = 1213.7^2$$
$$(x + 100)^2 + (y - 230)^2 = 576.6^2$$

Compute a Solution of the Model

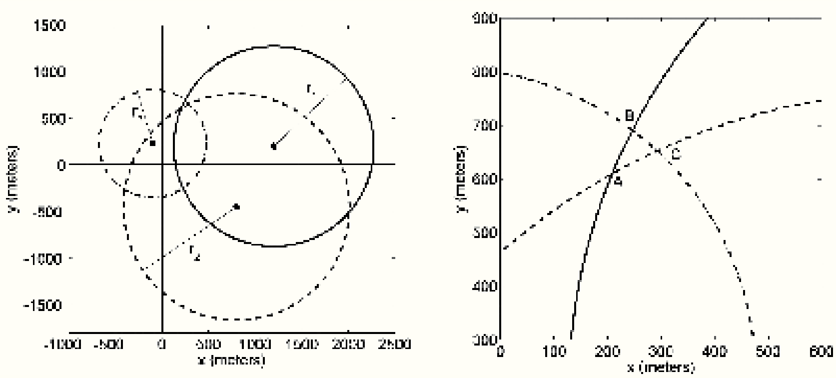
One way to find a solution is to find the intersection of two circles at a time. Each pair of circles has two intersections, so we have to be careful to choose the appropriate one. **Figure 3a** shows the circles and the location of the towers. **Figure 3b** shows a close-up of the region expected to contain the cell phone. The three circles do not intersect at a single point even though the points A, B, and C appear close to one another. The distance between points A and C is about 94 m, and the area of the triangular region is about 2600 m^2.

Interpret the Solution and Draw Conclusions

On the basis of the findings, we could select one of the three points as the location of the cell phone; however, none of the points is a more reliable location than the others. Alternatively, we can interpret the answer to mean that the cell phone is located in some region containing the points A, B, and C. We conclude that the cell phone is most likely in this region.

Validate Conclusions

We have concluded that the cell phone is in a 2600 m^2 triangular region, so now we must reflect on whether this is a sufficiently small area in the



$A = (208.8, 610)$
$B = (246.3, 691.1)$
$C = (292.4, 652.5)$

(a) (b) (c)

Fig. 3 The radius of each circle is the distance from its central tower to the lost cell phone (**a**). A region where the circles nearly intersect is shown enlarged; notice that the bounded region is outside the circle with radius r_2 (**b**). The coordinates of the points A, B, and C define a triangular region of about 2600 m^2 (**c**).

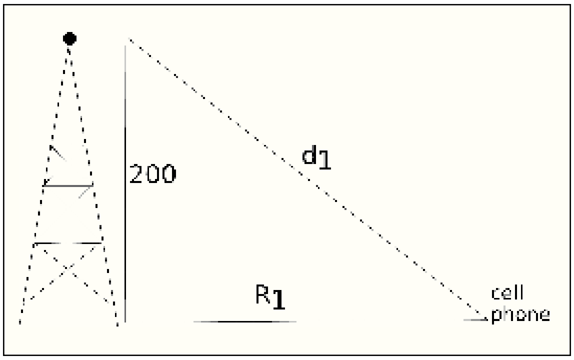


Fig. 4 This schematic shows a 200 m tower with a receiver at the top. The distance to the cell phone recorded by the tower is d_1. The distance from the phone to the base of the tower is R_1.

context of the problem. An area of about 2600 m^2 is about half a football field. In an urban space such as a shopping mall, this region may be too large to find a lost cell phone (or a missing person carrying a cell phone). How can we improve our model? Can we modify the model to give a more specific answer?

Develop and Formulate a New or Modified Model

What else do we know about cell towers that we have not taken into account? One consideration is that the signal receivers are at the top of the towers, not at the bottom. Therefore, the previous assumption of all distances being in the *xy*-plane should be replaced to account for the tower heights. The heights are not specified, so we make a new assumption that all towers have the same height—say, 200 meters (students can use other values based on research findings of average tower heights).

Using this new assumption, we develop a new or modified model. In this case, it is possible to modify the previous model by first computing the distances from the cell phone to the base of each tower (see **fig. 4**). The distance 1072.7 m from tower 1 to the cell phone is represented by d_1 in the figure. The horizontal distance is R_1 and can be computed using the Pythagorean theorem: $R_1^2 + 200^2 = 1072.7^2 \rightarrow R_1 = 1053.9$ m. Doing the same for the other towers, we get $R_2^2 + 200^2 = 1213.7^2 \rightarrow R_2 = 1197.1$ m and $R_3^2 + 200^2 = 576.6^2 \rightarrow R_3 = 540.8$ m.

With these new horizontal distances, the modified equations for (x, y) follow:

$$(x - 1200)^2 + (y - 200)^2 = 1053.9^2$$
$$(x - 800)^2 + (y + 450)^2 = 1197.1^2$$
$$(x + 100)^2 + (y - 230)^2 = 540.8^2$$

The new circles and new intersection points are shown in **figure 5**. Although it looks similar to the previous one, the region where the circles nearly intersect is about one-third smaller. The area of the new triangular region is about 819 m^2, and the distance between A and C is about 52 m. At this stage, students again interpret their solution, focus on the meaning of their new solution, and draw conclusions based on this smaller area that their model yielded. Students can now validate their outcome by making sense of their solution in the context of finding the lost cell phone.

Report the Solution

The cell phone is most likely located in the 819 m^2 triangular region with vertices given by the points $A = (224.9, 600)$, $B = (245.3, 646.3)$, and $C = (270.7, 623.7)$ in the city's coordinate system.

CONNECTIONS TO STANDARDS

Although "Model with mathematics" is one of CCSSM's Standards for Mathematical Practice (SMP 4; CCSSI 2010, p. 7), the cell phone problem shows how the modeling process naturally promotes students' engagement with the other mathematical practices. Within modeling, students must

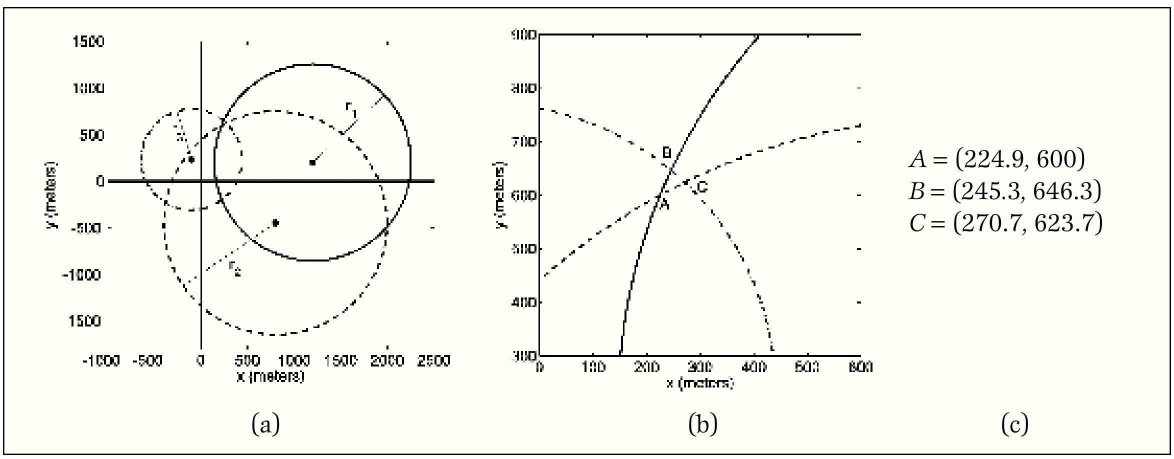


Fig. 5 The radius of each circle is calculated as the distance from the top of the central tower, 200 m high, to the lost cell phone at ground level (**a**). A close-up of the region shows where the circles nearly intersect (**b**). Coordinates of the points A, B, and C define a triangular region of about 819 m^2 (**c**).

make sense of the problem to propose assumptions, and because of the nature of the modeling cycle, they need to persevere (SMP 1; CCSSI 2010, p. 1) to find a satisfactory solution within the given context. Students must also "construct viable arguments and critique the reasoning of others" (SMP 3; CCSSI 2010, pp. 6–7) as they share solutions and justify choices in their model. Mathematical modeling provides an ideal setting for students to hold critical discussions and question one another about their decisions throughout the modeling process. This process in turn creates opportunities for students to provide justification, form conclusions, and explain their reasoning.

The Lost Cell Phone problem allows students to consider the mathematics that they know to address the situation. This particular problem can be approached using multiple concepts, including equations, variables, points, distance formula, right triangles, and intersection of multiple circles, which fall within the algebraic domain of reasoning with equations and inequalities (A-REI; CCSSI 2010) and the geometric domain of expressing geometric properties and equations (G-GPE; CCSSI 2010). We presented an initial model based on the equations of circles. The realization that the circles do not intersect at a single point can be verified graphically and algebraically and forces the students to think of a new set of assumptions or new approaches. Our revised model accounted for the tower heights using the Pythagorean theorem, which provides a concrete connection between algebra and geometry. However, students should be allowed to propose other assumptions and their own models, even if teachers do not anticipate their assumptions.

ASSESSMENT OF THE MODELING PROCESS

We developed general modeling categories (see **table 2**) and a rubric with criteria (see **table 3**) for evaluating modeling problems that take into consideration the modeling process, the model itself, the solution, and reflections on the students' process in creating a mathematical model. This general assessment rubric helps teachers evaluate the process that students take to produce quality work as they navigate through the modeling cycle and promotes their familiarity with the elements.

Define expectations clearly so that the evaluation process is transparent. The initial modeling assignments are critical so that students understand how their work will be evaluated. Getting students to write about mathematics is often challenging, but they become better writers and thinkers of mathematics when they have regular opportunities to write about engaging problems in mathematics (NCTM 2000). Modeling provides an excellent opportunity to promote this type of discourse.

The nature of mathematical modeling necessarily gives a sense of freedom for teachers to allow students to define their own problems with guidance. We argue that teaching this type of decision making and critical thinking is essential. Central to learning the structure of mathematical modeling are the initial understanding of the situation, identifying essential variables, making appropriate assumptions, and knowing the work that all stages of the modeling process entail. The modeling cycle provides a structure that students can follow explicitly to build a high level of understanding of the modeling process as they engage in the types of problems that lead to rich learning experiences.

Table 2 Mathematical Modeling Task Categories for Evaluating

Categories for Scoring	Score	Observations and Comments
Reasonable and relevant assumptions		
Formulation of a model based on assumptions		
Solution of the model		
Interpretation of solution		
Validation of conclusions		
Modeling cycle—revised assumptions and model		
Reflection on thought process as the model was created and the modeling cycle considered		
Reflection on other ways to model this problem besides the approach taken		
	Total	

Table 3 A General Rubric for Assessment of Modeling Problems						
	5	**4**	**3**	**2**	**1**	**0**
Explanations	Demonstrates full understanding and provides full explanations: justifications and explanations demonstrate understanding of concepts	Demonstrates basic understanding and provides minimal explanations	Demonstrates some understanding but has some gaps, brief explanations	Demonstrates little understanding, gaps in thinking, little to no explanations	Demonstrates some partial understanding with no explanation	Shows no evidence of understanding
Connections	Shows ideas that are well-connected: uses more than one concept and shows understanding of their connection	Shows ideas that are well connected	Shows ideas that are partially connected, missing points, or unclear	Shows ideas that are not well connected	Shows ideas that are not well connected	
Work	Provides complete work: includes assumptions and solutions that follow them	Provides complete work	Provides incomplete work but includes essential information	Provides incomplete work, missing some essential information	Provides incomplete work, missing essential information	Shows no evidence of work
Reasoning	Shows evidence of thoughtfulness and reasoning	Shows evidence of reasoning	Shows evidence of partial reasoning	Shows little evidence any reasoning	Shows evidence of faulty reasoning	
Concepts	Displays correct conceptual mathematical ideas: uses appropriate and multiple representations, uses appropriate concepts for the given problem	Displays correct conceptual mathematical ideas	Displays partial correct conceptual mathematical ideas	Displays incorrect conceptual mathematical ideas	Displays incorrect conceptual mathematical ideas	
Calculations	Presents correct calculations (possibly one minor error): numerical calculations are correct, appropriate units are used	Presents minor errors in calculations	Presents consequential errors in calculations	Presents significant errors in calculations	Presents significant errors in calculations	

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